Pliska Stud. Math. **29** (2018), 11–20

PLISKA
STUDIA MATHEMATICA

CRITICAL POINTS OF DIRAC FUNCTIONAL WITH BROKEN SYMMETRY

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In this paper, we prove the existence of a radially symmetric critical point of the Dirac functional with broken symmetry.

1. Introduction

In this paper, we shall be concerned with the Dirac equation

$$\imath \frac{\partial}{\partial t}\psi(t,\, x) + \mathcal{D}_m \psi(t,\, x) + V(x)\psi(t,\, x) = f(x,\, \psi(t,\, x)). \tag{1}$$

The unknown function ψ is defined on $(t,\, x) \in [0,\, T] \times \mathbf{R}^N$, for any $N \geq 3$, and takes value in $\mathbf{C}^d$, where $d := 2^{\lfloor (N+1)/2 \rfloor}$.

Definition 1. *The Dirac operator, denoted by $\mathcal{D}_m$, is the self-adjoint operator that is defined by setting*

$$\mathcal{D}_m := \sum_{j=1}^{N} \imath \Gamma^0 \Gamma^j \partial_j - m\Gamma^0,$$

where $\Gamma^0,\, \Gamma^1,\, \ldots,\, \Gamma^N$ are the $N+1$ generalized Dirac matrices, and ∂_j is the derivative with respect to the j-th space variable.

2010 *Mathematics Subject Classification:* 35J50, 35Q40, 35J45.
Key words: Dirac equation, variational methods, solitary waves.
*The author was supported in part by INDAM, GNAMPA – Gruppo Nazionale per l'Analisi Matematica, la Probabilita e le loro Applicazion, by Institute of Mathematics and Informatics, Bulgarian Academy of Sciences and by Top Global University Project, Waseda University.

Remark. In [3] it has been proved that the generalized Dirac matrices generate a representation of the real Clifford algebra $C\ell_{\alpha,\beta}(\mathbf{R})$ with parameters $\alpha = 1$ and $\beta = N - 1$.

Time-independent. In order to find a solution of (1), we consider the time-independent Dirac equation

$$\mathcal{D}_m\psi(x) + V(x)\psi(x) = f(x,\, \psi(x)), \qquad x \in \mathbf{R}^N,\ N \geq 3. \tag{2}$$

The main result is the following theorem, which asserts that, under certain assumptions, there exists a radially symmetric solution (in the weak sense) ψ of the equation (2).

Theorem 1. *Let $m > 0$ and consider the energy functional*

$$E(\psi) = \frac{1}{2}\,\mathrm{Re}\,\langle \mathcal{D}_m\psi,\, \psi\rangle_{L^2(\mathbf{R}^N)} + \langle V\psi,\, \psi\rangle_{L^2(\mathbf{R}^N)} - \int_{\mathbf{R}^N} F(x,\, \psi)\,\mathrm{d}x, \tag{3}$$

where $F(x,\, \psi) := \int_0^\psi f(x,\, s)\,\mathrm{d}s$. If $f : \mathbf{R}^n \times \mathbf{C}^d \longrightarrow \mathbf{C}^d$ and $V : \mathbf{R}^n \longrightarrow \mathbf{R}$ satisfy, respectively, the assumptions (F1)–(F4) *and* (V1)–(V3)*, with $\omega \in (0,\, m)$, then E admits a nonzero critical point*

$$\psi \in H^{1/2}_{\mathrm{rad}}\left(\mathbf{R}^N;\, \mathbf{C}^d\right).$$

Time-dependent. As a corollary of Theorem 1, we can immediately obtain an existence result concerning the equation (1).

More precisely, if φ denotes the solution given by Theorem 1, then one can check that $\psi(x,\, t) := \varphi(x)\,\mathrm{e}^{\imath\frac{2\pi}{T}t}$ is a periodic solution of the problem

$$\imath\frac{\partial}{\partial t}\psi(t,\, x) + \mathcal{D}_m\psi(t,\, x) + (V(x) + \frac{2\pi}{T})\psi(t,\, x) = f(x,\, \psi(t,\, x)), \tag{4}$$

provided that the nonlinear term satisfies the additional property:

$$f\left(x,\, \varphi_m(x)\,\mathrm{e}^{\imath\frac{2\pi}{T}t}\right) = f\left(x,\, \varphi_m(x)\right)\,\mathrm{e}^{\imath\frac{2\pi}{T}t}. \tag{5}$$

We also notice that the nonlinear term $f(x,\, \psi) = \psi(x)|\psi(x)|^{p-2}$, which is our model, satisfies this additional property.

Potential. Let $V : \mathbf{R}^n \longrightarrow \mathbf{R}$ be a potential satisfying (V1)–(V3). We may always assume, without loss of generality, that V is equal to zero. Indeed, let

$$\widetilde{\mathcal{D}}_m := \mathcal{D}_m + V.$$

The assumption (V3) states that 0 lies in a gap in the spectrum of the operator $\widetilde{\mathcal{D}}_m$, and therefore it admits a spectral decomposition, which is equal to the one introduced in the next Section for $\mathcal{D}_m$.

2. Variational Formulation of the Problem

The stationary solutions for a self-interacting Dirac field are usually introduced via critical points of the Dirac type functional (3).

Assumptions on f. We want to study the problem for a nonlinear term $f(x, \psi)$ similar to the model $\psi|\psi|^{p-2}$, which also satisfies (5).

Therefore we now introduce suitable assumptions on $f(x, \psi)$ in such a way as to have a similar behavior. More precisely, we require the following conditions to be satisfied:

(V1) The potential $V : \mathbf{R}^N \to \mathbf{C}^d$ is continuous and periodic of period 1 with respect to each spatial variable x_j, for $j = 1, \ldots, N$.

(V2) There exists a positive constant ω such that

$$0 \leq V(x) \leq \liminf_{|y| \to +\infty} V(y) = \omega < \infty,$$

and the inequalities are strict on a Borel subset E of nonzero Lebesgue measure.

(V3) The origin 0 lies in a gap of the spectrum of the operator $\widetilde{\mathcal{D}}_m = \mathcal{D}_m + \mathcal{V}$.

(F1) The function $f : \mathbf{R}^N \times \mathbf{C}^d \longrightarrow \mathbf{C}^d$ is continuous and periodic of period 1 with respect to each spatial variable x_j, for $j = 1, \ldots, N$.

(F2) There exist positive constants $c_1, c_2 > 0$ such that

$$|f(x, \psi)| \leq c_1 + c_2 \cdot |\psi|^{p-1}$$

for some $2 < p < 2^*$, where $2^* := \frac{2N}{N-2}$ is the Sobolev critical exponent.

(F3) There exists $\gamma > 2$ such that for any $x \in \mathbf{R}^N$ and any $\psi \neq 0$

$$0 < \gamma F(x,\, \psi) \leq |\psi| f(x,\, \psi).$$

(F4) The function f is superlinear, that is, for $|\psi| \to 0$ we have

$$f(x,\, \psi) = o\left(|\psi|\right).$$

Remark. Under these assumptions, the functional E is well-defined on

$$\mathcal{H} := H^{1/2}(\mathbf{R}^N;\ \mathbf{C}^d).$$

Remark. It follows from (F1), (F2) and (F4) that for every $\epsilon > 0$ there exists a positive constant $c_\epsilon > 0$ such that

$$|f(x,\, \psi)| \leq \epsilon|\psi| + c_\epsilon|\psi|^{p-1}.$$

The assumptions (V1) and (F1), on the other hand, imply the invariance of f and V under the action of $\mathbf{Z}^N$ on the space variables

We shall denote by $\|\cdot\|_{H^{1/2}}$ the norm of the space $\mathcal{H}$. The next result follows immediately from [7, Lemma 3.10].

Lemma 1. *Under the assumptions of Theorem* 1 *it turns out that E is a functional of class $C^1(\mathcal{H};\, \mathbf{R})$, and $\psi \in \mathcal{H}$ is a weak solution of* (2) *if and only if*

$$\mathrm{d}E(\psi)[\varphi] = 0, \qquad \forall\, \varphi \in \mathcal{H}.$$

2.1. Orthogonal Decomposition

The Dirac operator $\mathcal{D}_m$ admits a spectral decomposition and, in particular, it is given by the positive part and the negative part. More precisely, we set

$$\mathcal{D}_m = P_m - Q_m, \quad P_m = \frac{\sqrt{\mathcal{D}_m\mathcal{D}_m^*} + \mathcal{D}_m}{2} = \frac{|\mathcal{D}_m| + \mathcal{D}_m}{2}, \quad Q_m = \frac{|\mathcal{D}_m| - \mathcal{D}_m}{2}.$$

Therefore, the energy space $\mathcal{H}$ can be decomposed as

$$\mathcal{H} := E^+ \oplus E^-,$$

where

$$E^+ := \{\psi \in \mathcal{H} \mid Q_m(\psi) = 0\}, \quad E^+ := \{\psi \in \mathcal{H} \mid P_m(\psi) = 0\}.$$

The energy functional can be easily rewritten as follows:

$$E(\psi) = \frac{1}{2}\left[\|P_m\psi\|^2 - \|Q_m\psi\|^2\right] - \int_{\mathbf{R}^N} F(x,\, \psi)\, \mathrm{d}x, \tag{6}$$

where $\|\cdot\|$ denotes the norm induced on $\mathcal{H}$ by the spectral decomposition, and it is clearly equivalent to $\|\cdot\|_{H^{1/2}}$.

3. Linking Method

In this section, we set the ground to prove the existence of a critical value for the time-independent functional $E(\psi)$ using topological means (linking results.)

We shall follow the paper [2] and generalize the notion of linking accordingly to the Leray-Schauder's topological degree (see [8, Chapter 2].)

Setting. We consider the two manifolds

$$\mathcal{C} := \left\{\psi \in E^+ \mid \|\psi\| = \rho\right\} = E^+ \cap S_\rho(\mathcal{H}),$$

and

$$\mathcal{M} := \left\{\psi \in \mathcal{H} \;\middle|\; \psi = w + \lambda e,\ w \in E^-,\ \|w\| \leq R,\ 0 \leq \lambda \leq \|e\|\right\},$$

where ρ and R are positive real numbers, and e is a fixed point of E^+ such that $\|e\| > \rho$. Moreover, we denote by Σ the collection of all the continuous homotopies $h \in C^0\left([0,\,1] \times \mathcal{H};\ \mathcal{H}\right)$ such that

$$Q \circ h(t,\, u) = Q(u) - W(t,\, u), \quad h(0,\, \cdot) = \mathrm{id}_H$$

where W_t is a compact perturbation for every fixed $t \in [0,\,1]$.

Definition 2. (Link) *Let $\mathcal{C}$, $\mathcal{M}$ and Σ be as above. We say that $\partial\mathcal{M}$ and $\mathcal{C}$ link if and only if for every $h \in \Sigma$ satisfying the property*

$$\mathcal{C} \cap h_t\left(\partial\mathcal{M}\right) = \emptyset, \qquad \forall\, t \in [0,\,1],$$

it turns out that

$$\mathcal{C} \cap h_t\left(\mathcal{M}\right) \neq \emptyset, \qquad \forall\, t \in [0,\,1].$$

3.1. Application of the Linking Method

In this brief paragraph, we prove that $\mathcal{C}$ and $\partial\mathcal{M}$ link in the sense of Definition 2, and we also show that the functional E behaves as expected.

Lemma 2. *For every positive constants ρ, $R > 0$, and for every point $e \in E^+$ such that $\|e\| > \rho$, the manifolds $\partial\mathcal{M}$ and $\mathcal{C}$ link.*

Proof. Let us denote by B_δ the closed ball of radius $\delta > 0$ and center the origin. As stated above, we have

$$\mathcal{C} = E^+ \cap \partial B_\rho.$$

On the other hand, by assumption $\|e\| > \rho$; hence it is not difficult to prove that there is an isomorphism

$$\mathcal{M} \cong \{\lambda e \mid \lambda \in [0, \|e\|]\} \oplus \left\{w \in E^- \mid \|w\| \leq R\right\}.$$

In particular, it turns out that

$$\mathcal{M} \cong \{\lambda e \mid \lambda \in [0, \|e\|]\} \oplus \left(E^- \cap B_R\right)$$

In conclusion, we apply [2, Lemma 1.2 and Lemma 1.3] with $E_1 := E^+$, $E_2 := E^-$, $S = \mathcal{C}$, $Q = \mathcal{M}$, $r := \lambda$, $r_1 := \|e\|$ and $r_2 := R$, and infer that $\mathcal{C}$ and $\partial\mathcal{M}$ link. □

3.2. Functional Properties

Following [1, Chapter 8, Section 3], we want to prove that the energy functional E satisfies the following properties:

(J.1) There are positive real numbers α, $\rho > 0$ such that

$$E(\psi) \geq \alpha, \qquad \forall\, \psi \in E^+ \cap S_\rho(\mathcal{H}) = \mathcal{C},$$

that is, the functional E is bounded from below on the manifold $\mathcal{C}$.

(J.2) There are positive real numbers β, $R > 0$ and a vector $e \in E^+$ such that $\|e\| > \rho$, $\alpha > \beta$ strictly, and

$$E(\psi) \leq \beta, \qquad \forall\, \psi \in \partial\mathcal{M}.$$

Theorem 2. *Under the same assumptions of Theorem 1, there exists $\rho_0 > 0$ such that* **(J.1)** *holds true for every $0 < \rho \leq \rho_0$.*

P r o o f. Let ψ be a function in $\mathcal{C}$. By definition, we have

$$E(\psi) = \frac{1}{2}\|\psi\|^2 - \int_{\mathbf{R}^N} F(x,\, \psi)\, \mathrm{d}x.$$

The first term is strictly positive; hence it is enough to find a bound on the second term that depends on ρ, and choose ρ_0 in such a way that $E(\psi)$ is bounded from below by 0. As mentioned in the previous Section, for every $\epsilon > 0$ there exists a positive constant $c_\epsilon > 0$ such that

$$|F(x,\, \psi)| \le \frac{\epsilon}{2}|\psi|^2 + \frac{c_\epsilon}{p+1}|\psi|^{p+1}.$$

If we take the integral with respect to the space variables x_j, then we obtain the following estimate:

$$\int_{\mathbf{R}^N} |F(x,\, \psi)|\; \mathrm{d}x \le \frac{\epsilon}{2}\, \|\psi\|^2_{L^2(\mathbf{R}^N)} + \frac{c_\epsilon}{p+1}\, \|\psi\|^{p+1}_{L^{p+1}(\mathbf{R}^N)}.$$

By assumption $2 < p < 2^*$; hence the Sobolev embedding theorem (see, e.g., [4]) implies that there exist positive constants $c_1,\, c_2 > 0$ such that

$$\int_{\mathbf{R}^N} |F(x,\, \psi)|\; \mathrm{d}x \le c_1\frac{\epsilon}{2}\, \|\psi\|^2_{H^{1/2}(\mathbf{R}^N)} + c_2\frac{c_\epsilon}{p+1}\, \|\psi\|^{p+1}_{H^{1/2}(\mathbf{R}^N)},$$

and, by the arbitrariness of $\epsilon > 0$, we infer that

$$\int_{\mathbf{R}^N} |F(x,\, \psi)|\; \mathrm{d}x = o\left(\|\psi\|^2_{H^{1/2}(\mathbf{R}^N)}\right).$$

Recall that the norm $\|\cdot\|$ is equivalent to the $\|\cdot\|_{H^{1/2}(\mathbf{R}^N)}$-norm by construction; hence it turns out that

$$\int_{\mathbf{R}^N} |F(x,\, \psi)|\; \mathrm{d}x = o\left(\|\psi\|^2\right),$$

as $\|\psi\| \to 0$. Consequently, there exists $\delta > 0$ small enough such that

$$\|\psi\| \le \delta \Rightarrow \int_{\mathbf{R}^N} |F(x,\, \psi)|\; \mathrm{d}x \le \frac{1}{4}\|\psi\|^2.$$

Therefore, if we take $\rho_0 := \delta$, then for any $\rho \in (0,\, \rho_0]$ the property **(J.1)** is satisfied (as a consequence of the previous estimate) with $\alpha > 0$ strictly. □

Theorem 3. *Under the same assumptions of Theorem 1, there exists $R_0 > 0$ such that* **(J.2)** *holds true for every $R \in (R_0, +\infty)$.*

Proof. By definition, the boundary of the manifold $\mathcal{M}$ is the disjoint union of three connected components, namely

$$\partial\mathcal{M}_1 = S_R(E^-) \oplus \{\lambda e \,:\, \lambda \in [0,\, 1]\},$$

and

$$\partial\mathcal{M}_2 = B_R(E^-), \quad \partial\mathcal{M}_3 = B_R(E^-) \oplus \{e\}.$$

By (F3), for every $\epsilon > 0$ there exists a positive constant $k_\epsilon > 0$ such that

$$F(x,\, \psi) \geq k_\epsilon |\psi|^\gamma - \epsilon|\psi|^2,$$

for every $x \in \mathbf{R}^N$ and $\psi \in \mathbf{C}^d$. If $\psi \in \partial\mathcal{M}$, then the functional E is given by

$$E(\psi) = \frac{1}{2}\left[\lambda^2 - \|w\|^2\right] - \int_{\mathbf{R}^N} F(x,\, w + \lambda e)\,\mathrm{d}x$$

and, clearly, we have the estimate

$$E(\psi) \leq \frac{1}{2}\left[\lambda^2 - \|w\|^2\right] + \epsilon\|w + \lambda e\|^2_{L^2(\mathbf{R}^N)} - k_\epsilon\|w + \lambda e\|^\gamma_{L^\gamma(\mathbf{R}^N)}.$$

By the Sobolev embedding theorem, there exists a positive constant $c > 0$ such that

$$E(\psi) \leq \frac{1}{2}\left[\lambda^2 - \|w\|^2\right] + c\epsilon\|w + \lambda e\|^2 - k_\epsilon\|w + \lambda e\|^\gamma_{L^\gamma(\mathbf{R}^N)}.$$

The decomposition $\mathcal{H} = E^+ \oplus E^-$ is orthogonal; hence

$$E(\psi) \leq \frac{1}{2}\left[\lambda^2 - \|w\|^2\right] + c\epsilon\left(\lambda^2 - \|w\|^2\right) - k_\epsilon\|w + \lambda e\|^\gamma_{L^\gamma(\mathbf{R}^N)}.$$

If we take $\epsilon > 0$ such that $c \cdot \epsilon \leq 1/4$, then it turns out that

$$E(\psi) \leq \frac{3}{4}\left[\lambda^2 - \|w\|^2\right] - k_\epsilon\|w + \lambda e\|^\gamma_{L^\gamma(\mathbf{R}^N)}.$$

We are finally ready to conclude the proof. First, we notice that

$$E(\psi) \to -\infty \quad \text{as} \quad \|w + \lambda e\| \to +\infty,$$

and thus it follows immediately that, if we choose $R > \rho_0 > 0$ big enough, then

$$E(\psi) \leq 0, \qquad \forall\, \psi \in \partial\mathcal{M}_1.$$

If $\psi \in \partial\mathcal{M}_2$, then $\psi = w \in E^-$ and

$$E(\psi) \leq -\frac{1}{2}\|w\|^2 - k_\epsilon \|w + \lambda e\|^\gamma_{L^\gamma(\mathbf{R}^N)} < 0,$$

and the inequality is strict (i.e., E is strictly negative on the second component of the boundary).

Finally, if $\psi \in \partial\mathcal{M}_3$, we can choose $e \in E^+$ such that $\|e\| = R$ and $E(e) = 0$, and we can argue as in [9, Proposition 2]. □

4. Existence of a Critical Point

Let us consider

$$c := \inf_{h \in \Sigma} \left[\sup_{\psi \in \mathcal{M}} E \circ h(\psi) \right]. \tag{7}$$

In Theorem 2 we proved that $\partial\mathcal{M}$ and $\mathcal{C}$ link; therefore it is easy to infer that $c \geq \rho > 0$ (see, e.g., [1, Theorem 8.22]).

In particular, if we prove that c is a critical value for E, then we automatically get a nonzero critical point $\psi \in \mathcal{H}$ for E, which is enough to conclude the proof of Theorem 1.

Recall that, a straightforward application of the interpolation inequality (Riesz-Thorin) and [6, Theorem 1], for any $N \geq 3$ and $p \in (2, 2^*)$, the following embedding is continuous and compact:

$$H^{1/2}_{\text{rad}}\left(\mathbf{R}^N\right) \hookrightarrow L^p\left(\mathbf{R}^N\right).$$

The same duality argument presented in the paper [5, Chapter 3] shows the existence of a Palais-Smale sequence $(u_n) \subset \mathcal{H}$ at the level c.

The invariance of f with respect to the group of translations $\mathbf{Z}^N$ implies that there exists a Palais-Smale sequence $(v_n) \subset \mathcal{H}$ at the level c such that v_n converges to $v \in \mathcal{H}$ weakly in $\mathcal{H}$.

On the other hand, the compactness of the embedding stated above also implies that v_n converges to v strongly in $L^2_{loc}(\mathbf{R}^N)$, which means that v is a weak solution to the equation (2), and this completes the proof of Theorem 1.

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