

GORENSTEIN STABLE GODEAUX SURFACES

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ABSTRACT. We classify Gorenstein stable numerical Godeaux surfaces with worse than canonical singularities and compute their fundamental groups.

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1. INTRODUCTION

Surfaces of general type with the smallest possible invariants, namely $K_X^2 = 1$ and $p_g(X) = q(X) = 0$ are called (numerical) Godeaux surfaces. For a Godeaux surface X with at most canonical singularities, the algebraic fundamental group $\pi_1^{\text{alg}}(X)$ is known to be cyclic of order $d \leq 5$ (see [Miy76, Rei78]). Hence $\pi_1^{\text{alg}}(X)$ and the torsion subgroup $T(X)$ of $\text{Pic}(X)$ are finite abelian groups dual to each other. The cases $d = 3, 4, 5$ are completely understood [Rei78, CU16], and quite a few examples of Godeaux surfaces with torsion $\mathbb{Z}/2$ or trivial have been constructed since then¹.

In this paper we classify non-classical Gorenstein stable Godeaux surfaces, i.e., Gorenstein stable surface X with $K_X^2 = 1$ and $p_g(X) = q(X) = 0$ which have worse than canonical singularities, and calculate their (topological) fundamental groups.

Denote by $\mathfrak{M}_{1,1}$ the Gieseker moduli space of *classical* Godeaux surfaces and by $\overline{\mathfrak{M}}_{1,1}$ its compactification, the moduli space of stable Godeaux surface (see [Kol12, Kol16] and references therein). The space $\mathfrak{M}_{1,1}$ is conjectured to have exactly 5 irreducible components, one for each possible $\pi_1^{\text{alg}}(X)$.

Some of the examples we construct turn out to be smoothable, that is, they are in the closure of $\mathfrak{M}_{1,1}$, some are known not to be smoothable and they make up

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¹See e. g. [Bar84, Bar85, CD89, Ino94, Wer94, DW99, DW01, LP07, RTU17, Cou].

TABLE 1. List of non-classical Gorenstein stable Godeaux surfaces

$ \pi_1(X) $	case	normal	smoothable	reference
5	(R)	✓	✓	Section 2.A
	(E ₅) general	—	✓	Section 3.D
	X _{1.5}	—	✓	Section 3.C
4	(R)	✓	✓	Section 2.A
	(B ₁)	✓	✓	Section 2.A
	(P ₁)	—	✓	Section 3.C
	(E ₄) general	—	✓	Section 3.D
3	(R)	✓	✓	Section 2.A
	(B ₂)	✓	✓	Section 2.A
	(P ₃)	—	✓	Section 3.C
	(E ₃) general	—	✓	Section 3.D
2	(E ₂) general	—	unknown	Section 3.D
	(P ₂)	—	unknown	Section 3.C
1	(E ₁)	—	unknown	Section 3.D
	(E ₂), reducible polarisation	—	unknown	Section 3.D
	(E _m), reducible pol., $m \geq 3$	—	✓ (Rem. 4.6)	Section 3.D
	(dP)	—	no [Rol16]	Section 3.B

an irreducible component of $\overline{\mathfrak{M}}_{1,1}$. The smoothability of the examples with small fundamental group is still to be investigated. So $\overline{\mathfrak{M}}_{1,1}$ has at least six irreducible components but might still turn out to be connected.

Instead of giving a detailed description of the single cases, we collect an overview of our results in Table 1. The statements made in the table and some further applications will be stated and proved in Section 4; the explicit constructions can be found in the sections indicated in the table.

2. NORMAL NON-CANONICAL STABLE GODEAUX SURFACES

2.A. Classification of possible cases. In this section we study normal Gorenstein stable Godeaux surfaces that are “non-canonical”, namely they have worse than canonical singularities. We first refine the results in [FPR15b, §4] in this particular case.

Proposition 2.1 — *Let X be a normal Gorenstein stable surface with $\chi(X) = K_X^2 = 1$ with worse than canonical singularities. Let $\varepsilon: \tilde{X} \rightarrow X$ be the minimal desingularisation.*

Then X has precisely one elliptic singularity and the exceptional divisor D of ε is a smooth elliptic curve; in addition, one of the following occurs:

- (B) *There exists a bi-elliptic surface X_{\min} , an irreducible divisor D_{\min} on X_{\min} and a point P such that:*
 - $D_{\min}^2 = 2$ and $P \in D_{\min}$ has multiplicity 2
 - \tilde{X} is the blow-up of X_{\min} at P
 - X is obtained from \tilde{X} by blowing down the strict transform of D_{\min} .
- (R) *\tilde{X} is a ruled surface with $\chi(\tilde{X}) = 0$.*

We first prove:

Lemma 2.2 — *Under the assumptions of Proposition 2.1, one has:*

- (i) $q(\tilde{X}) = 1$ and $p_g(\tilde{X}) = 0$, that is, $\chi(\tilde{X}) = 0$;
- (ii) X has exactly one elliptic singularity and the corresponding exceptional divisor is a smooth elliptic curve.

Proof. By the classification of Gorenstein log-canonical surface singularities, the singularities of X are either canonical or elliptic Gorenstein and by [FPR15a, Cor. 4.2] $\chi(X) = 1 > 0$ implies $q(X) = p_g(X) = 0$. So we write $\varepsilon^*K_X = K_{\tilde{X}} + \tilde{D}_1 + \cdots + \tilde{D}_k$, with the \tilde{D}_i disjoint 2-connected effective divisors with $p_a(\tilde{D}_i) = 1$. We write $\tilde{D} = \tilde{D}_1 + \cdots + \tilde{D}_k$.

Since $k > 0$ by assumption, by [FPR15b, Lem. 4.2] we have $\chi(\tilde{X}) = \chi(X) - k = 1 - k \leq 0$, hence $q(\tilde{X}) > 0$. Again by [FPR15b, Lem. 4.2] we also have $p_g(\tilde{X}) \leq h^0(K_{\tilde{X}} + \tilde{D}) = p_g(X) = 0$; it follows that the Albanese image of \tilde{X} is a curve B . For every $i = 1, \dots, k$ the standard restriction sequence induces an injection $\mathbb{C} \simeq H^0(K_{\tilde{D}_i}) \rightarrow H^1(K_{\tilde{X}})$, hence no \tilde{D}_i is contained in a fibre of the Albanese map of \tilde{X} . It follows that B has genus 1, and therefore $q(\tilde{X}) = 1$, $\chi(\tilde{X}) = 0$, $k = 1$ and $\tilde{D} = \tilde{D}_1$ is a smooth elliptic curve. \square

Proof of Proposition 2.1. By Lemma 2.2, by [FPR15b, Thm. 4.1] and by the classification of surfaces, it is enough to exclude that \tilde{X} is a minimal properly elliptic surface and that the exceptional divisor \tilde{D} of the minimal resolution $\varepsilon: \tilde{X} \rightarrow X$ is a smooth curve of genus 1 with $\tilde{D}^2 = -1$.

So assume by contradiction that this is the case. Since $c_2(\tilde{X}) = 0$ by Noether's formula, applying the formula for computing the topological Euler characteristic ([BHPV04, Prop. III.11.4]) to the elliptic fibration $f: \tilde{X} \rightarrow B$ one sees that all fibres of f have smooth support, namely f is a *quasi-bundle* (cf. [Ser93], [Ser96]). So the surface \tilde{X} is a free quotient $(F \times C)/G$ where:

- F is a curve of genus 1, C is a curve of genus $g > 1$
- G is a finite group that acts faithfully on F and C ; we let G act diagonally on $F \times C$
- the elliptic fibration is induced by the second projection $p_2: F \times C \rightarrow C$.

Let now $\Gamma \subset F \times C$ be an irreducible component of the preimage of \tilde{D} : the map $\Gamma \rightarrow \tilde{D}$ is étale, and so Γ is an elliptic curve and is mapped to a point by p_2 . It follows that \tilde{D} is mapped to a point by f , hence $\tilde{D}^2 = 0$, against the assumptions. \square

The surfaces of type (B) can be described more explicitly. We start by giving two examples.

Example 2.3 — By Proposition 2.1, a normal Gorenstein surface of type (B) is determined by a pair (X_{\min}, D_{\min}) where X_{\min} is a minimal bi-elliptic surface and D_{\min} is an irreducible divisor with $D_{\min}^2 = 2$ that has a double point P . We describe two types of such pairs:

- (B₁) Let E be an elliptic curve, set $A := E \times E$ and let generators $e_1, e_2 \in G \cong (\mathbb{Z}/2)^2$ act on A as follows:

$$(x, y) \xrightarrow{e_1} (x + \tau_1, y + \tau_1); \quad (x, y) \xrightarrow{e_2} (x + \tau_2, -y),$$

where τ_1 and τ_2 generate $E[2]$. The group G acts freely on A and the surface $X_{\min} := A/G$ is bi-elliptic.

The diagonal $\Delta \subset A$ is e_1 -invariant and it intersects $e_2\Delta$ transversally at 4 points. The divisor $\Delta + e_2\Delta$ is G -invariant and its image D_{\min} in X_{\min} is an irreducible curve with $D_{\min}^2 = 2$ that has a node P .

(B₂) Let $\zeta := e^{\frac{2\pi i}{3}}$, denote by E the elliptic curve $\mathbb{C}/(\mathbb{Z} + \mathbb{Z}\zeta)$ and denote by ρ the order 3 automorphism of E induced by multiplication by ζ . We let $A := E \times E$ and consider the following automorphisms of A :

$$(x, y) \xrightarrow{e_1} (x + \tau_1, y + \tau_1); \quad (x, y) \xrightarrow{e_2} (x + \tau_2, \rho y),$$

where $\tau_1 = \frac{1-\zeta}{3}$ and τ_1 and τ_2 generate $E[3]$. Since $\rho(\tau_1) = \tau_1$, the automorphisms e_1 and e_2 commute, and they generate a subgroup G of $\text{Aut}(A)$ isomorphic to $(\mathbb{Z}/3)^2$. The group G acts freely on A and the surface $X_{\min} := A/G$ is bi-elliptic.

The diagonal $\Delta \subset A$ is e_1 -invariant and it intersects $e_2\Delta$ and $(2e_2)\Delta$ transversally at 3 points. By symmetry, $e_2\Delta$ and $(2e_2)\Delta$ also intersect transversally at 3 points. So the divisor $Z := \Delta + e_2\Delta + (2e_2)\Delta$ is G -invariant and $Z^2 = 18$. It follows that the image D_{\min} of Z in X_{\min} is irreducible with $D_{\min}^2 = 2$. The group G acts transitively on the nodes of Z , so D_{\min} is a curve of arithmetic genus 2 with one node. The group $\text{Aut}(E)$ is cyclic of order 6, generated by the automorphism ρ induced by multiplication by $-\zeta$. It is immediate to check that $\rho(\tau_1) = -\tau_1$, so that the subgroup $\langle \tau_1 \rangle$ is preserved by $\text{Aut}(E)$, and that the remaining six elements of $E[3]$ form an orbit.

Proposition 2.4 — *Let X be a normal Gorenstein Godeaux surface of type (B). Then X is obtained as in case (B₁) or (B₂) of Example 2.3.*

Proof. Let (X_{\min}, D_{\min}) be the pair giving the surface X as in Proposition 2.1. The surface X_{\min} , being bi-elliptic, is a quotient $(B \times C)/G$, where B and C are elliptic curves, G is a finite abelian group that acts faithfully on B and C and:

- the free action of G on $B \times C$ is the diagonal action
- B/G is elliptic and C/G is rational.

Denote by Γ the preimage of D_{\min} in $B \times C$ and write $\Gamma = \Gamma_1 + \dots + \Gamma_r$, with the Γ_i irreducible. The normalisation of Γ is an étale cover of the exceptional divisor \tilde{D} of the minimal desingularisation $\varepsilon: \tilde{X} \rightarrow X$, hence the Γ_i have geometric genus 1; since $B \times C$ is an abelian variety, it follows that the Γ_i are smooth elliptic with $\Gamma_i^2 = 0$.

Notice that by construction the set of the Γ_i is a G -orbit and that $r = [G : H]$, where H is the stabilizer of Γ_1 . We claim that all elements of H act on C as translations. Indeed, assume by contradiction that $h \in H$ is not a translation. Then, denoting by σ a generator of $H^0(\omega_C)$, we have $h^*\sigma = \lambda\sigma$ for some $1 \neq \lambda \in \mathbb{C}$. On the other hand, the action of H on Γ_1 is free, since G acts freely on A , and therefore we have $h^*\tau = \tau$, where τ is a generator of $H^0(\omega_{\Gamma_1})$. So, denoting by $p_2: B \times C \rightarrow C$ the second projection, we conclude that $p_2^*\sigma$ restricts to zero on Γ_1 , and therefore Γ_1 is a fibre of p_2 . This implies that D_{\min} is contracted by the map $X_{\min} \rightarrow C/G$, which is impossible since $D_{\min}^2 = 2 > 0$.

By standard computations, $H^{1,1}(X_{\min})$ is 2-dimensional, generated by the classes of the images \bar{B} and \bar{C} of $B \times \{0\}$ and $\{0\} \times C$, hence D_{\min} is numerically a linear combination with rational coefficients of \bar{B} and \bar{C} . Pulling back to $B \times C$, one sees

that Γ is numerically equivalent to $\beta(B \times \{0\}) + \gamma(\{0\} \times C)$ for some $\beta, \gamma \in \mathbb{Q}$. By construction, $\Gamma^2 = 2 \cdot |G|$ since $D_{\min}^2 = 2$, hence $|G| = \frac{1}{2}\Gamma^2 = \beta\gamma$. Moreover, since the intersection numbers $\Gamma_i(B \times \{0\})$ and $\Gamma_i(\{0\} \times C)$ are independent of i , we have $\beta = r(\Gamma_1(\{0\} \times C))$ and $\gamma = r(\Gamma_1(B \times \{0\}))$. Hence the order of G is divisible by $[G : H]^2$. Since H consists of translations, this remark rules out five of the seven cases of the Bagnera-de Franchis list (cf. [Bea83, List VI.20]) and we are left with the following possibilities:

$$\begin{aligned} G &= \mathbb{Z}/2 \times \mathbb{Z}/2 \text{ and } r = 2, \\ G &= \mathbb{Z}/3 \times \mathbb{Z}/3 \text{ and } r = 3. \end{aligned}$$

In both cases, by the above computations the Γ_i project isomorphically onto B and C , hence B and C are isomorphic and we may identify B and C in such a way that, say, Γ_1 is the diagonal. Hence we have either case (B_1) or (B_2) \square

2.B. Computation of fundamental groups. We devote the rest of the section to the description of the fundamental group and thus of the torsion group of normal non-canonical Gorenstein stable Godeaux surfaces. Our results, more precisely Proposition 2.8, Lemma 2.11, and Proposition 2.12, are summarised in the following:

Theorem 2.5 — *Let X be a normal non-canonical Gorenstein Godeaux surface. Then:*

- (i) *If X is of type (B_1) then $\pi_1(X)$ and $T(X)$ are cyclic of order 4.*
- (ii) *If X is of type (B_2) then $\pi_1(X)$ and $T(X)$ are cyclic of order 3.*
- (iii) *If X is of type (R) , then $\pi_1(X)$ and $T(X)$ are cyclic of order $3 \leq d \leq 5$.*

We deal first with surfaces of type (B) and start with two preparatory lemmas.

Lemma 2.6 — *Let X be a normal Godeaux surface of type (B), obtained from a pair (X_{\min}, D_{\min}) and let G be a finite group of order d . Then the connected étale G -covers $Y \rightarrow X$ are naturally in one-to-one correspondence with the connected étale G -covers $Y_{\min} \rightarrow X_{\min}$ such that the preimage of D_{\min} in Y_{\min} has d irreducible components.*

Proof. The correspondence goes as follows: if $Y_{\min} \rightarrow X_{\min}$ is an étale cover as in the statement, its pull-back $\tilde{Y} \rightarrow \tilde{X}$ is an étale G -cover whose restriction to the exceptional divisor \tilde{D} is the trivial cover. If $\tilde{Y} \rightarrow Y \rightarrow X$ is the Stein factorisation of the induced morphism $\tilde{Y} \rightarrow X$ then $Y \rightarrow X$ is a connected étale G -cover.

The inverse correspondence is described in an analogous way: if $Y \rightarrow X$ is an étale G -cover and $\tilde{Y} \rightarrow \tilde{X}$ is the induced étale cover of \tilde{X} , then the exceptional curve E of the blow up $\tilde{X} \rightarrow X_{\min}$ pulls back to a sum $E_1 + \cdots + E_d$ of disjoint (-1) -curves of \tilde{Y} . Contracting each E_i to a point yields an étale cover $Y_{\min} \rightarrow X_{\min}$. \square

Lemma 2.7 — *Let $f: Y \rightarrow X$ be a birational morphism of normal surfaces and let E_1, \dots, E_n be the connected components of the exceptional locus with inclusions $\iota_j: E_j \hookrightarrow Y$. If $\pi_1(Y)$ is abelian then*

$$\pi_1(X) = \frac{\pi_1(Y)}{\langle \text{im } \iota_{j*} \mid j = 1, \dots, n \rangle}$$

Proof. Fix an arbitrary base point in Y . A priori the image of $\pi_1(E_j)$ in $\pi_1(X)$ is determined only up to conjugation, but in our situation it is uniquely determined

since $\pi_1(X)$ is abelian. The result then follows by induction from Seifert–van Kampen applied to the homotopy-equivalent model provided by [FPR15a, Prop. 3.1]. \square

Proposition 2.8 — *Let X be a normal Godeaux surface of type (B) . Then:*

- (i) *If X is of type (B_1) then $\pi_1(X)$ and $T(X)$ are cyclic of order 4.*
- (ii) *If X is of type (B_2) then $\pi_1(X)$ and $T(X)$ are cyclic of order 3.*

Proof. (i) We will directly construct the universal cover of X , starting with a degree 4 cover of X_{\min} which is trivial on the normalisation of D_{\min} , as suggested by Lemma 2.6. We continue to use the notation introduced in Example 2.3 and let $E_1 := E/\langle\tau_1\rangle$. Fix in addition an element η_2 such that $2\eta_2 = \tau_2$ and consider the map

$$\Phi: E \times E \rightarrow E \times E, \quad (x, y) \mapsto (x + y + \eta_2, x - y + \eta_2)$$

which is a degree 4 isogeny of $E \times E$ composed with the translation by (η_2, η_2) .

A direct computation shows that Φ induces the map Ψ in the following diagram

$$\begin{array}{ccc} E \times E & \xrightarrow{4:1} & E_1 \times E_1 \\ \Phi \downarrow 4:1 & & \Psi \downarrow 2:1 \\ E \times E & \xrightarrow{/e_1} & (E \times E)/\langle e_1 \rangle \xrightarrow{2:1} X_{\min} \end{array} \quad \begin{array}{c} \searrow \Pi \\ \cdot \end{array}$$

Now consider the automorphism σ of $E \times E$ given by $\sigma(x, y) = (y + \tau_2, x)$. It descends to an automorphism $\bar{\sigma}$ of $E_1 \times E_1$ of order 4 which satisfies $\Psi \circ \bar{\sigma} = \bar{e}_2 \circ \Psi$, where \bar{e}_2 is the induced action on $(E \times E)/\langle e_1 \rangle$. Hence $X_{\min} = (E_1 \times E_1)/\langle \bar{\sigma} \rangle$ and Π is an étale $\mathbb{Z}/4$ -cover.

Recall that by construction the pullback Γ of D_{\min} to $E \times E$ is the union of $\Delta = \{x - y = 0\}$ and $e_2\Delta = \{x + y + \tau_2 = 0\}$. Hence $\Phi^*\Gamma$ is the union for $\tau \in E[2]$ of the four curves $E \times \{\tau\}$ and the four curves $\{\tau\} \times E$. Thus

$$\Pi^*D_{\min} = E_1 \times \{0\} + E_1 \times \{\bar{\tau}_2\} + \{0\} \times E_1 + \{\bar{\tau}_2\} \times E_1,$$

where $\bar{\tau}_2$ is the image τ_2 in E_1 .

Now let $\tilde{Y} \rightarrow Y_{\min} := E_1 \times E_1$ be the blow up in the four nodes of Π^*D_{\min} and $q: \tilde{Y} \rightarrow Y$ the contraction of the four elliptic curves with self-intersection -2 . This results in a diagram

$$(2.9) \quad \begin{array}{ccccc} & & \tilde{Y} & & \\ & \swarrow & \downarrow & \searrow q & \\ Y_{\min} & & \tilde{X} & & Y \\ & \swarrow & \downarrow & \searrow & \downarrow \\ X_{\min} & & X & & X \end{array}$$

where all vertical arrows are étale $\mathbb{Z}/4$ -covers. By Lemma 2.7 applied to q the surface Y is simply connected, since $\pi_1(\tilde{Y}) = \pi_1(Y_{\min}) = \pi_1(E_1 \times \{0\}) \times \pi_1(\{0\} \times E_1)$, and thus Y is the universal cover of X and $\pi_1(X) = \mathbb{Z}/4$.

(ii) As in the proof of (i) we use the notation of Example 2.3. We set $Y_{\min} = (E \times E)/\langle e_1 \rangle$ and let $\Pi: Y_{\min} \rightarrow X_{\min}$ be the quotient map for the automorphism \bar{e}_2 of Y_{\min} induced by e_2 . By construction Π^*D_{\min} is the orbit of the image of the diagonal under the action of \bar{e}_2 and as such consists of the three components $\bar{\Delta}$, $\bar{e}_2\bar{\Delta}$, and $2\bar{e}_2\bar{\Delta}$, which intersect pairwise in one node.

Blowing up the nodes of Π^*D_{\min} and contracting the elliptic curves we get a diagram as in (2.9) where now the vertical maps are étale $\mathbb{Z}/3$ -covers. To conclude, it remains to show that Y is simply connected, or equivalently, by Lemma 2.7, that $\pi_1(Y_{\min})$ is generated by $\pi_1(\bar{\Delta})$, $\pi_1(\bar{e}_2\bar{\Delta})$ and $\pi_1(2\bar{e}_2\bar{\Delta})$. Since all groups involved are abelian, we work in integral homology.

Identifying $H_1(E \times E) = \mathbb{Z}[\zeta]^2 \subset \mathbb{C}^2$ we find as subgroups of \mathbb{C}^2 :

$$\begin{aligned} H_1(Y_{\min}) &= \langle (1, 0), (\zeta, 0), (0, 1), (0, \zeta), (\tau_1, \tau_1) \rangle_{\mathbb{Z}}, \\ H_1(\bar{\Delta}) &= \langle (1, 1), (\zeta, \zeta), (\tau_1, \tau_1) \rangle_{\mathbb{Z}}, \end{aligned}$$

and because \bar{e}_2 acts in homology by multiplication with ζ in the second variable, $\zeta\tau_1 = \tau_1 + \zeta$ and $\zeta^2 + \zeta + 1 = 0$,

$$\begin{aligned} H_1(\bar{e}_2\bar{\Delta}) &= \langle (1, \zeta), (\zeta, \zeta^2), (\tau_1, \tau_1 + \zeta) \rangle_{\mathbb{Z}}, \\ H_1(2\bar{e}_2\bar{\Delta}) &= \langle (1, \zeta^2), (\zeta, 1), (\tau_1, \tau_1 - 1) \rangle_{\mathbb{Z}}. \end{aligned}$$

Thus the subgroup generated by the three sub-lattices contains the element $(0, 1) = (\tau_1, \tau_1) - (\tau_1, \tau_1 - 1)$ and then we easily find all generators of $H_1(Y_{\min})$, so that $\pi_1(Y)$ is trivial. \square

Before we address stable Godeaux surfaces of type (R) we state a general result that was proved by Reid in the classical case.

Lemma 2.10 — *Let X be a Gorenstein stable Godeaux surface, possibly non-normal. Then X has no étale $(\mathbb{Z}/2)^2$ -cover.*

Proof. We argue by contradiction, following the argument used in [Rei78]. Assume $p: Y \rightarrow X$ is an étale $(\mathbb{Z}/2)^2$ -cover and let $\{0, \eta_1, \eta_2, \eta_3\}$ be the kernel of $p^*: \text{Pic}(X) \rightarrow \text{Pic}(Y)$. Since $h^2(K_X + \eta_i) = h^0(\eta_i) = 0$, $h^0(K_X + \eta_i) \geq \chi(K_X) = 1$ and we may find non-zero sections $\sigma_i \in H^0(K_X + \eta_i)$, $i = 1, 2, 3$. The curves D_i defined by the σ_i are irreducible and meet each other transversally at one point, since $D_i D_j = K_X D_i = 1$ and K_X is ample. We have $\sigma_i^2 \in H^0(2K_X)$, for $i = 1, 2, 3$. Since $P_2(X) = 2$ by Riemann–Roch [LR16, Thm. 3.1] we have a relation $\sum \lambda_i \sigma_i^2 = 0$. By this relation, the D_i all have a common point P , which is smooth for each of them. It follows that $\eta_3|_{D_3} = (\eta_1 - \eta_2)|_{D_3} = (D_1 - D_2)|_{D_3} = 0$. Consider a desingularisation $\varepsilon: \tilde{X} \rightarrow X$ and let $Z \rightarrow \tilde{X}$ be the (connected!) double cover given by $\varepsilon^* \eta_3$: then the preimage in Z of $\varepsilon^* D_3$ has two connected components with self-intersection equal to 1, contradicting the Index Theorem. \square

Lemma 2.11 — *Let X be a normal Godeaux surface of type (R). Then $\pi_1(X)$ is a finite abelian group of order $d \geq 3$.*

Proof. As usual, let $\varepsilon: \tilde{X} \rightarrow X$ be the minimal resolution and let \tilde{D} be the exceptional divisor. Denote by $p: \tilde{X} \rightarrow B$ the Albanese map, let F be a general fibre of p and let $d = F\tilde{D}$, so that the induced map $a: \tilde{D} \rightarrow B$ is a degree d isogeny. Then by Lemma 2.7 we have

$$\pi_1(X) \cong \frac{\pi_1(\tilde{X})}{\pi_1(\tilde{D})} \cong \frac{\pi_1(B)}{a_* \pi_1(\tilde{D})},$$

which is a finite abelian group of order d .

Since $K_{\tilde{X}}F = -2$ and $K_{\tilde{X}} + \tilde{D}$ is ample on F , we have $0 < F(K_{\tilde{X}} + \tilde{D}) = d - 2$, and we get $d \geq 3$. \square

Proposition 2.12 — *Let X be a normal Godeaux surface of type (R) . Then $\pi_1(X)$ is cyclic of order $d \leq 5$.*

Remark 2.13 — An example of type (R) with $T(X)$ of order 5 appears in [Lee00, Ex. 2.14]. It is constructed by specializing the general construction of Godeaux surfaces with torsion $\mathbb{Z}/5$, hence it is smoothable. An example with $T(X)$ of order 4 appears in [MP16, Example (1) in §6.3]. Again, this is constructed as a degeneration of smooth Godeaux surfaces with $\mathbb{Z}/4$ torsion.

Proof. By Lemma 2.11 the group $\pi_1(X)$ is finite abelian of order $d \geq 3$ and by Lemma 2.10 it is sufficient to prove that $d \leq 5$. Consider the connected étale cover $Y \rightarrow X$ with Galois group $G = \pi_1(X)$, let $f: \tilde{Y} \rightarrow \tilde{X}$ be the induced cover, set $\Gamma := f^*\tilde{D}$ and $M := f^*(K_{\tilde{X}} + \tilde{D}) = K_{\tilde{Y}} + \Gamma$. In fact we have seen in the proof of Lemma 2.11 that $\tilde{Y} = Y \times_B \tilde{D}$, where $\tilde{D} \rightarrow B$ is induced by the Albanese map, thus $\Gamma = \Gamma_1 + \dots + \Gamma_d$, where the Γ_i are smooth disjoint sections of the Albanese map with $\Gamma_i^2 = -2$. Moreover, we have $M^2 = d$ since $(K_{\tilde{X}} + \tilde{D})^2 = 1$.

For the reader's convenience we break the proof into steps:

Step 1: $h^0(M) = d - 1 \geq 2$ and the curves Γ_i are not contained in the fixed part of $|M|$.

For $i = 1, \dots, d$, consider the adjunction sequence:

$$0 \rightarrow K_{\tilde{Y}} \rightarrow K_{\tilde{Y}} + \Gamma_i \rightarrow K_{\Gamma_i} = \mathcal{O}_{\Gamma_i} \rightarrow 0.$$

The coboundary map $H^0(K_{\Gamma_i}) \rightarrow H^1(K_{\tilde{Y}})$ is dual to $H^1(\mathcal{O}_{\tilde{Y}}) \rightarrow H^1(\mathcal{O}_{\Gamma_i})$, hence it is an isomorphism, since Γ_i is a section of the Albanese map. The claim now follows from the long exact sequence in cohomology of

$$0 \rightarrow K_{\tilde{Y}} \rightarrow M = K_{\tilde{Y}} + \Gamma \rightarrow \bigoplus_{i=1}^d K_{\Gamma_i} \rightarrow 0.$$

Step 2: $|M|$ has no fixed part.

Assume by contradiction that $|M| = |P| + Z$, with $Z > 0$. The group G acts on $|M|$ by construction, hence Z is a G -invariant divisor. In addition, since G is abelian, $H^0(M)$ splits as a direct sum of eigenspaces under the G -action, and so there exist a G -invariant divisor $M_0 = P_0 + Z \in |M|$. It follows that P_0 is also invariant; the image C_0 of M_0 in X is a curve in $|K_X + \eta|$ for some $\eta \in T(X)$. Since the only curves contracted by the map $\tilde{Y} \rightarrow \tilde{X} \rightarrow X$ are the Γ_i , no component of Z is contracted by this map by Step 1 and not every component of P_0 is contracted because P_0 moves. So, C_0 is reducible with $K_X C_0 = 1$, a contradiction since K_X is ample.

Step 3: If $d \geq 4$, then $|M|$ has no base points.

The base locus of $|M|$ is finite by Step 2. Assume by contradiction that $|M|$ has a base point Q . Then, because G acts on $|M|$, all the d points in the orbit of Q are base points of $|M|$. Since $M^2 = d$ and M moves, we conclude that $|M|$ has precisely d simple base points, and so it is a pencil, namely $d - 1 = 2$ by Step 1.

Step 4: $d \leq 5$.

Assume by contradiction $d \geq 6$. Consider the morphism $h: \tilde{Y} \rightarrow \mathbb{P}^{d-2}$ given by $|M|$ and denote by Σ the image of h , which is a surface because $|M|$ has no base points by Step 3 and $M^2 > 0$. Since a non-degenerate irreducible surface in \mathbb{P}^k has degree $\geq k - 1$, we have:

$$d = M^2 \geq \deg \Sigma \deg h \geq (d - 3) \deg h,$$

hence either $\deg h = 1$, or $d = 6$ and $\deg h = 2$.

Assume $\deg h = 1$. We have $MK_{\tilde{Y}} = M^2 - M\Gamma = d - 0 = d$, hence by the adjunction formula the general $M \in |M|$ is smooth of genus $d + 1$. The morphism

h maps M birationally to a curve of degree d in \mathbb{P}^{d-3} . So Castelnuovo's bound [ACGH85, III, §2] gives $g(M) \leq 4$ if $d = 6$ and $g(M) \leq 3$ if $d \geq 7$, and we have reached a contradiction.

Finally assume $\deg h = 2$ and $d = 6$. In this case $\Sigma \subset \mathbb{P}^4$ is either a rational normal scroll or the cone over the twisted cubic (see e.g. [EH87]), and therefore it has a unique ruling by lines. Denote by W the moving part of the pullback of this ruling to \tilde{Y} and let Φ be a general divisor in W . W is a G -invariant pencil and it is not composed with the Albanese map. Indeed, if F is a fibre of the Albanese map of \tilde{Y} , then arguing as in the proof of Lemma 2.11 we get $FM = d - 2 = 4$, hence F is mapped to a curve of degree ≥ 2 . Since the Albanese pencil is the only pencil of rational curves of \tilde{Y} , it follows that the general Φ is connected of positive genus. We have $M\Phi = 2$ by construction, hence by the Index Theorem $\Phi^2 \leq \frac{(M\Phi)^2}{M^2} < 1$ and therefore $\Phi^2 = 0$ and $K_{\tilde{Y}}\Phi \geq 0$. Since $2 = M\Phi = (K_{\tilde{Y}} + \Gamma)\Phi$, by parity we either have $K_{\tilde{Y}}\Phi = 0$, $\Gamma\Phi = 2$ or $K_{\tilde{Y}}\Phi = 2$, $\Gamma\Phi = 0$. On the other hand, $\Gamma\Phi$ is divisible by 6, because the Galois group G acts transitively on the set of components $\Gamma_1, \dots, \Gamma_6$ of Γ and W is G -invariant, so we have $\Gamma\Phi = 0$, $K_{\tilde{Y}}\Phi = 2$ and W is a free pencil of genus 2 curves stable under a free action of a group of order 6. We are going to show that this is not possible, arguing as in the proof of [MP06, Lem. 2.2] (see also [CMP07]).

Let $\psi: \tilde{Y} \rightarrow \mathbb{P}^1$ be the map given by W . The action of G descends to an action on \mathbb{P}^1 : denote by $H_0 < G$ the subgroup of elements acting trivially on \mathbb{P}^1 , let $S_0 := \tilde{Y}/H_0$ be the quotient surface, $p_0: S_0 \rightarrow \mathbb{P}^1$ the morphism induced by ψ and Φ' the general fibre of p_0 . Since H_0 acts freely, the surface S_0 is smooth and we have $|H_0|(K_{S_0}\Phi') = K_{\tilde{Y}}\Phi = 2$. Since $K_{S_0}\Phi'$ is even by adjunction, the only possibility is that H_0 is the trivial subgroup. Let now $H < G$ be the subgroup of order 3 and again set $S := \tilde{Y}/H$, denote by $p: S \rightarrow \mathbb{P}^1$ the induced morphism and Φ'' the general fibre of p . Since H acts faithfully on \mathbb{P}^1 , the curve Φ'' has genus 2 and p has a triple fiber over the two fixed points of H on \mathbb{P}^1 , contradicting the adjunction formula. \square

3. NON-NORMAL STABLE GODEAUX SURFACES

3.A. Normalisation and glueing: starting point of the classification. We will now show how a combination of the results of [FPR15b] and Kollár's glueing principle leads to a classification of non-normal Gorenstein stable Godeaux surfaces.

Let X be a non-normal stable surface and $\pi: \bar{X} \rightarrow X$ its normalisation. Recall that the non-normal locus $D \subset X$ and its preimage $\bar{D} \subset \bar{X}$ are pure of codimension 1, i.e. curves. Since X has ordinary double points at the generic points of D the map on normalisations $\bar{D}^\nu \rightarrow D^\nu$ is the quotient by an involution τ . Kollár's glueing principle says that X can be uniquely reconstructed from $(\bar{X}, \bar{D}, \tau: \bar{D}^\nu \rightarrow D^\nu)$ via the following two push-out squares:

$$(3.1) \quad \begin{array}{ccccc} \bar{X} & \xleftarrow{\bar{\iota}} & \bar{D} & \xleftarrow{\bar{\nu}} & \bar{D}^\nu \\ \downarrow \pi & & \downarrow \pi & & \downarrow / \tau \\ X & \xleftarrow{\iota} & D & \xleftarrow{\nu} & D^\nu \end{array}$$

Applying this principle to non-normal Gorenstein stable Godeaux surfaces, we deduce by [Kol13, Thm. 5.13] and [FPR15b, Addendum in Sect.3.1.2] that a triple

(\bar{X}, \bar{D}, τ) corresponds to a Gorenstein stable Godeaux surface if and only if the following four conditions are satisfied:

lc pair condition: (\bar{X}, \bar{D}) is an lc pair, such that $K_{\bar{X}} + \bar{D}$ is an ample Cartier divisor.

$K_{\bar{X}}^2$ -condition: $(K_{\bar{X}} + \bar{D})^2 = 1$.

Gorenstein-glueing condition: $\tau: \bar{D}^\nu \rightarrow \bar{D}^\nu$ is an involution that restricts to a fixed-point free involution on the preimages of the nodes of \bar{D} .

χ -condition: The holomorphic Euler-characteristic of the non-normal locus D is $\chi(D) = 1 - \chi(\bar{X}) + \chi(\bar{D})$.

In [FPR15b] we classified lc pairs (\bar{X}, \bar{D}) (with $\bar{D} \neq 0$) satisfying the first and second condition, and also showed (Thm. 3.6 loc. cit.) that an involution τ such that (\bar{X}, \bar{D}, τ) satisfies all four conditions can only exist in three cases, labelled (dP) , (P) , and (E_+) .

We will now classify the possible involutions τ for (P) and (E_+) , and investigate the geometry and topology of the resulting Gorenstein stable Godeaux surfaces; the case (dP) has been treated in [Rol16] and we recall the results in Section 3.B.

3.B. Case (dP) . In this case \bar{X} is a del Pezzo surface of degree 1 (possibly with canonical singularities) and $\bar{D} \in |-2K_{\bar{X}}|$ is a nodal curve. If there exists an involution τ such that (\bar{X}, \bar{D}, τ) gives rise to a Gorenstein stable Godeaux surface X then [FPR15b, Lem. 3.5] implies that \bar{D} is the union of two nodal anti-canonical curves and that the non-normal locus $D \subset X$ has arithmetic genus 2. Thus there is a unique possibility for the glueing and we get the following:

Proposition 3.2 — *If X is a Gorenstein stable Godeaux surface with normalisation a del Pezzo surface of degree 1, then X is as described in [Rol16]. In particular, X is simply connected and not smoothable.*

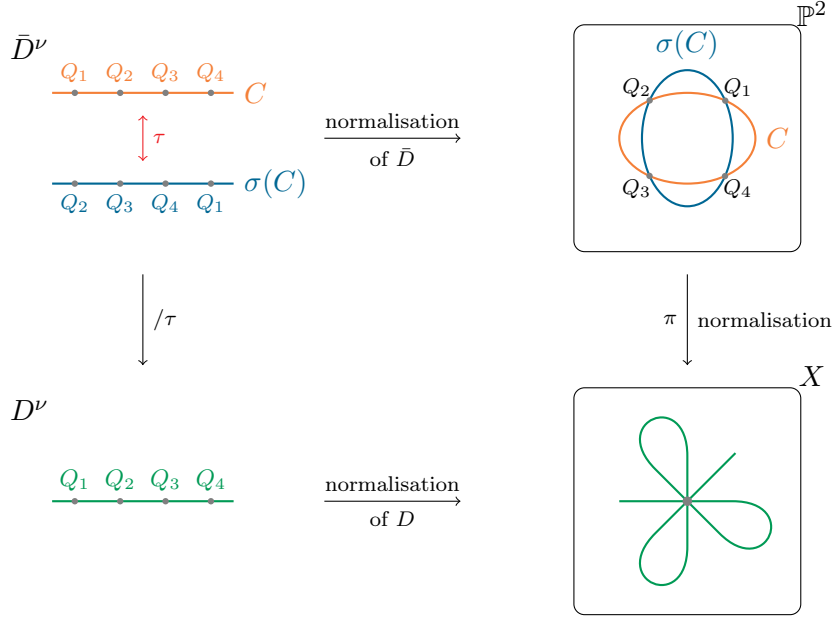
3.C. Case (P) . In this case $\bar{X} = \mathbb{P}^2$ and \bar{D} can be any nodal plane quartic, so $p_a(\bar{D}) = 3$ and the χ -condition is satisfied if and only if $p_a(D) = 3$.

We fix some notation first and then we write down three constructions, each depending on one parameter.

Fix $Q_1, \dots, Q_4 \in \mathbb{P}^2$ points in general position and denote by \mathcal{C} the pencil of conics through these points. Denote by L_{ij} the line through Q_i and Q_j ; the conics $L_{12} + L_{34}$, $L_{13} + L_{24}$ and $L_{14} + L_{23}$ are the only reducible conics of \mathcal{C} . Any permutation $\gamma \in S_4$ of Q_1, \dots, Q_4 determines an automorphism of \mathbb{P}^2 that in turn induces an automorphism of \mathcal{C} ; by abuse of notation we still denote all these automorphisms by the same letter. The action of S_4 on \mathcal{C} is not faithful: indeed a permutation induces the identity on \mathcal{C} if and only if it fixes the three reducible conics. So the kernel of the map $S_4 \rightarrow \text{Aut}(\mathcal{C})$ is the subgroup $H = \{\text{id}, (12)(34), (13)(24), (14)(23)\}$ and therefore the image of S_4 is a subgroup of $\text{Aut}(\mathcal{C})$ isomorphic to S_3 .

Remark 3.3 — If $C \in \mathcal{C}$ is a smooth member, then the cross-ratio $\beta_C \in \mathbb{C} - \{0, 1\}$ of the points Q_1, Q_2, Q_3, Q_4 is well defined. It is a classical fact (cf. for instance [FFP16, Es. 4.24]) that β_C determines $C \in \mathcal{C}$. We denote by $j(C) := \frac{(\beta_C^2 - \beta_C + 1)^3}{\beta_C^2(\beta_C - 1)^2}$ the j -invariant of the unordered quadruple $\{Q_1, \dots, Q_4\}$ on $C \cong \mathbb{P}^1$. So we have a rational function $j: \mathcal{C} \rightarrow \mathbb{P}^1$ such that the restriction of j to the set of irreducible elements of \mathcal{C} is the quotient map for the S_4 -action.

From this one can easily deduce that different general choices of parameter in the constructions below give non-isomorphic surfaces.


 FIGURE 1. The general surface in (P_1)

Case (P_1) : Let $\sigma = (1234) \in S_4$. The permutation σ acts on \mathcal{C} as an involution that fixes $L_{13} + L_{24}$ and a smooth conic C_0 . Since $L_{12} + L_{34}$ and $L_{14} + L_{23}$ are switched by σ , the three singular conics and C_0 are a harmonic quadruple of \mathcal{C} . Given $C \in \mathcal{C}$ such that $C \neq \sigma(C)$, we set $\bar{D} := C + \sigma(C)$ and we take as τ the involution of $\bar{D}^\nu = C \sqcup \sigma(C)$ that identifies C with $\sigma(C)$ via the restriction of σ . Clearly τ satisfies the Gorenstein condition, and the stable surface thus obtained has a degenerate cusp at the image point of Q_1, \dots, Q_4 . If C is irreducible, then the double curve D is a rational curve with a unique (semi-normal) quadruple point, hence $p_a(D) = 3$ and the χ -condition is satisfied. We give a graphical representation in Figure 1.

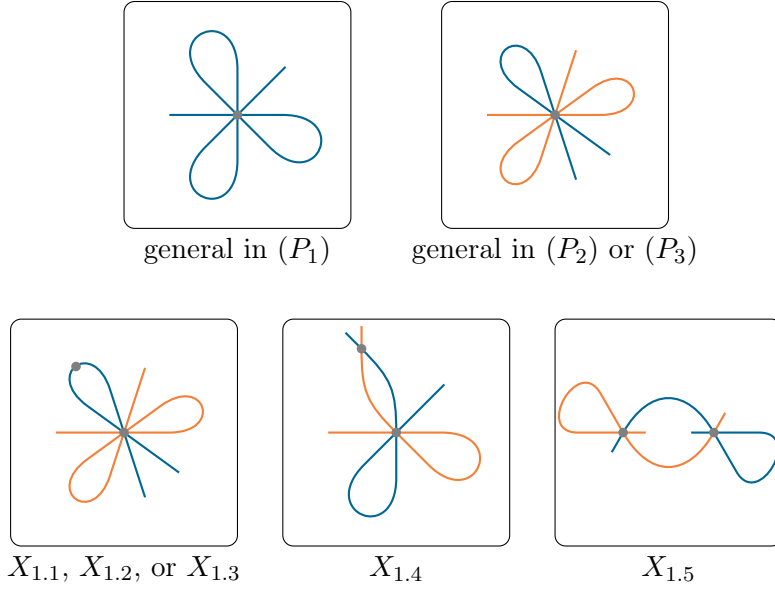
If $C = L_{12} + L_{34}$ is reducible, the double locus splits in two components, the quadruple point persists and an additional node appears, so we have $p_a(D) = 3$ also in this case. In the classification of surfaces with normalisation $(\mathbb{P}^2, 4 \text{ lines})$ (see [FPR15a], Table 1) such a configuration occurs only once, in the case $X_{1,4}$.

Case (P_2) : Set $\rho = \sigma^2 = (13)(24) \in S_4$. The action of ρ on \mathcal{C} is trivial, namely ρ preserves the conics of \mathcal{C} . We take $\bar{D} = C + L_{13} + L_{24}$, with $C \in \mathcal{C}$ distinct from $L_{13} + L_{24}$. The involution τ on $\bar{D}^\nu = C \sqcup L_{13} \sqcup L_{24}$ is defined on C as the restriction of ρ ; in addition, τ switches L_{13} with L_{24} , identifying them via the only isomorphism such that:

$$R \mapsto Q_2, \quad Q_1 \mapsto Q_4, \quad Q_3 \mapsto R',$$

where R denotes the point $L_{13} \cap L_{24}$ if we consider it on L_{13} , and R' denotes the same point if we consider it on L_{24} ; the involution τ satisfies the Gorenstein condition.

If C is irreducible, then the double locus D of the corresponding stable surface has two components; the image of C , which is a nodal rational curve, and the image of L_{13} and L_{24} , which is a rational curve with a unique (semi-normal) triple point; the components meet at the singular points, which combine to a semi-normal quintuple point. In particular, $p_a(D) = 3$ and X is a stable Godeaux surface.

FIGURE 2. Non-normal loci in case (P)

When C is reducible, then the double curve remains the same but the surface develops an additional degenerate cusp on the nodal components of the non-normal locus, which locally is isomorphic to a cone over a nodal plane cubic, and we have a stable Godeaux surface also in this case. More precisely, if $C = L_{14} + L_{23}$, then the Gorenstein condition is still verified and one can check that the resulting stable Godeaux surface is case $X_{1.1}$ in Table 1 of [FPR15a], while if $C = L_{12} + L_{34}$ then we get case $X_{1.2}$.

Case (P_3) : We take \bar{D} as in case (P_2) , but we choose a different involution: we let τ be the involution that acts on $L_{13} \sqcup L_{24}$ as in case (P_2) and on C via the permutation (12)(34). The difference to the previous family is that here the involution on C does not preserve the intersection $L_{13} \cap C$. When C is smooth the non-normal locus looks like in case (P_2) , but the way it is glued into the surfaces, encoded by the involution, is different.

If $C = L_{14} + L_{23}$, then the Gorenstein condition is satisfied and we get case $X_{1.3}$ in Table 1 of [FPR15a].

If $C = L_{12} + L_{34}$, then the Gorenstein condition is not satisfied, because τ fixes the preimages in $L_{12} \sqcup L_{34}$ of the singular point of C . The non-normal locus has 3 components and at the newly developed node X has a non-Gorenstein singularity of index 2.

Proposition 3.4 — *Let X be a Gorenstein stable surface with normalisation \bar{X} isomorphic to \mathbb{P}^2 . Then either X is obtained as in one of the above cases (P_1) , (P_2) or (P_3) , or it is isomorphic to the surface $X_{1.5}$ in Table 1 of [FPR15a].*

Proof. Let μ_1 be the number of degenerate cusps in X , let ρ be the number of ramification points of the map $\bar{D}^\nu \rightarrow D^\nu$ and let $\bar{\mu}$ be the number of nodes of \bar{D} . Then by [FPR15b, Lem. 3.5] we have the equality $\chi(D) = \frac{1}{2}(\chi(\bar{D}) - \bar{\mu}) + \frac{\rho}{4} + \mu_1$. Using $\chi(D) = \chi(\bar{D}) = -2$ we obtain

$$(3.5) \quad \bar{\mu} = \frac{\rho}{2} + 2\mu_1 + 2$$

Thus \bar{D} has at least one node, which implies $\mu_1 \geq 1$ by the classification of Gorenstein slc singularities (see the proof of Lemma 3.5 in [FPR15b]). Also, a plane quartic can have at most 6 nodes, so in total we get $4 \leq \bar{\mu} \leq 6$.

Note that a plane quartic with at least 4 nodes is reducible, so \bar{D} consists of four lines, two lines and a conic, two conics, or a line and a nodal cubic. We proceed case by case.

$\bar{D} = \text{four general lines}$: This case has been classified in [FPR15a, Sect. 4.2]. The four surfaces $X_{1.1}$, $X_{1.2}$, $X_{1.3}$ and $X_{1.4}$ appear as special cases of construction (P_1) , (P_2) or (P_3) and $X_{1.5}$ is listed separately.

$\bar{D} = \text{a conic and two lines}$: The gluing involution τ has to preserve the conic and exchange the two lines. By the Gorenstein condition, the involution on the conic cannot fix any of the four intersection points with the pair of lines.

By (3.5) we have a unique degenerate cusp in X , thus τ cannot interchange the preimages in \bar{D}' of the intersection of the two lines.

Now there are two possibilities: either the involution on the conic preserves the intersection with each individual line and we have case (P_2) , or it does not and we have case (P_3) .

$\bar{D} = \text{two irreducible conics}$: By (3.5) we have a unique degenerate cusp and τ has no fixed points on \bar{D}' . Thus τ exchanges the two conics C and C' , that is, we have an abstract isomorphism $\varphi = \tau|_C: C \rightarrow C'$ preserving the four intersection points Q_1, \dots, Q_4 . We can number the intersection points such that $\varphi(Q_i) = Q_{i+1}$ (where $Q_5 = Q_1$) because otherwise there would be more than one degenerate cusp.

Now consider the unique automorphism σ of \mathbb{P}^2 that acts on the Q_i in the same way as φ and let σC be the image of C under σ . The composition $\sigma \circ \varphi^{-1}: C' \rightarrow \sigma C$ is an abstract isomorphism of two plane conics fixing four points in the plane. By [FFP16, Es. 4.24] (see also Remark 3.3) it is actually induced by the identity on \mathbb{P}^2 , thus $C' = \sigma C$ and $\varphi = \sigma|_C$ and we are in case (P_1) .

$\bar{D} = \text{a nodal cubic and a line}$: The involution has to preserve the line, because the number of marked points on the two components of \bar{D}' is different. In particular any τ fixes the preimage of a node of \bar{D} and the Gorenstein condition cannot be satisfied.

We have enumerated all possible cases for \bar{D} and thus concluded the classification. \square

3.D. Case (E_+) .

3.D.1. *Stable Godeaux surfaces and bi-tri-elliptic curves.* Let X be a stable Godeaux surface such that its normalisation \bar{X} is the symmetric product of an elliptic curve E . We continue to use the notation from (3.1). Recall that the conductor locus $\bar{D} \subset \bar{X}$ is a stable curve of arithmetic genus two, which is a trisection of the Albanese map $a: \bar{X} = S^2E \rightarrow E$. The χ -condition (cf. §3.A) implies that $\chi(D) = \chi(X) - \chi(\bar{X}) + \chi(\bar{D}) = 0$, so we get a configuration

$$(3.6) \quad \begin{array}{ccc} & \bar{D} & \\ \pi \swarrow & & \searrow q \\ D & & E \end{array},$$

$\begin{array}{ccc} & & \\ & 2:1 & \\ & & \\ & & 3:1 \end{array}$

where D and E are smooth elliptic curves and q is the restriction of the Albanese map to \bar{D} . This motivates the following definition (cf. [FFP16]).

Definition 3.7 — A bi-tri-elliptic configuration is a diagram of curves as in (3.6) where D, E are elliptic curves, \bar{D} is a stable curve of genus 2 and q and π are finite with $\deg q = 3$ and $\deg \pi = 2$. The map π is the quotient by an involution which we call the bi-elliptic involution of the configuration.

A stable curve of genus 2 is called bi-tri-elliptic curve if it admits a bi-tri-elliptic configuration.

It turns out that this configuration of curves determines the surface X uniquely.

Proposition 3.8 — *There is a bijection (up to isomorphism) between bi-tri-elliptic configurations and Gorenstein stable Godeaux surfaces with normalisation the symmetric product of an elliptic curve.*

Proof. We have seen above that every Gorenstein stable Godeaux surfaces with normalisation the symmetric product of an elliptic curve gives rise to a uniquely determined bi-tri-elliptic configuration. We now need to show that we can recover the surface X just from the datum (3.6). The main step is

Lemma 3.9 — *Let $q: \bar{D} \rightarrow E$ be a triple cover, with E an elliptic curve and \bar{D} an at most nodal curve of genus 2. Then \bar{D} embeds in S^2E as a trisection of the Albanese map $S^2E \rightarrow E$.*

Proof. The trace map gives a splitting $q_*\mathcal{O}_{\bar{D}} = \mathcal{O}_E \oplus \mathcal{E}^\vee$, where \mathcal{E} is a rank 2 vector bundle on E . By [CE96, Thm 1.3] (cf. also, [Mir85]), the curve \bar{D} is embedded in $\mathbb{P}(\mathcal{E})$ as a trisection and $\chi(\mathcal{E}^\vee) = \chi(\mathcal{O}_{\bar{D}}) = -1$, so by Riemann–Roch $\deg \mathcal{E} = 1$. The adjunction formula for a finite morphism [Har77, Chap. III, Ex. 7.2] gives

$$q_*\omega_{\bar{D}} = \mathcal{H}om_{\mathcal{O}_E}(q_*\mathcal{O}_{\bar{D}}, \omega_E) = \omega_E \oplus \omega_E \otimes \mathcal{E} = \mathcal{O}_E \oplus \mathcal{E}$$

and in particular $h^0(\mathcal{E}) = 1$; analogously, given a point $P \in E$, we get $0 = h^0(\omega_{\bar{D}}(-q^*P)) = h^0(\mathcal{O}_E(-P)) + h^0(\mathcal{E}(-P))$, hence $h^0(\mathcal{E}(-P)) = 0$. It follows that the only section of \mathcal{E} vanishes nowhere and it gives an exact sequence $0 \rightarrow \mathcal{O}_E \rightarrow \mathcal{E} \rightarrow \mathcal{L} \rightarrow 0$, where \mathcal{L} is a line bundle of degree 1. The sequence is not split, since $h^0(\mathcal{E}) = 1$. This proves that $\mathbb{P}(\mathcal{E}) \cong S^2E$, see e. g. [CC93, §1]. \square

Write $\bar{X} = S^2E$ and let Φ be a fibre of the Albanese map $a: \bar{X} \rightarrow E$; since $h^0(\mathcal{E}) = 1$, there is a section C_0 of a with $C_0^2 = 1$. The classes of C_0 and Φ generate $\text{Pic}(\bar{X})$ up to numerical equivalence and $K_{\bar{X}}$ is algebraically equivalent to $-2C_0 + \Phi$. Since $\bar{D}\Phi = 3$ and $2 = (K_{\bar{X}} + \bar{D})\bar{D}$ by adjunction, it follows that \bar{D} is algebraically equivalent to $3C_0 - \Phi$ and $K_{\bar{X}} + \bar{D}$ is algebraically equivalent to C_0 .

Denote by τ the bi-elliptic involution of the bi-tri-elliptic configuration. Then, by Kollár’s glueing principle, the triple (\bar{X}, \bar{D}, τ) gives rise to a stable surface X . The Gorenstein condition and the χ -condition (cf. §3.A) are easily checked and we get a Gorenstein stable Godeaux surface.

Clearly, the two constructions are inverse to each other, up to isomorphism. \square

Thus to understand Gorenstein stable Godeaux with normalisation a symmetric product of an elliptic curve we need to classify bi-tri-elliptic configurations.

3.D.2. *Classification of bi-tri-elliptic curves.* We now recall the classification of bi-tri-elliptic configurations from [FPR16], where we study more generally (p, d) -elliptic configurations on curves of genus 2.

If \bar{D} is a stable bi-tri-elliptic curve, then it is either smooth or the union of two elliptic curves meeting at a point. So the Jacobian $A := J(\bar{D}) = \text{Pic}^0(\bar{D})$ is compact, that is, \bar{D} is of compact type, and \bar{D} can be embedded in A as a principal polarisation.

Following work of Frey and Kani (see [FK91]), the Jacobian of a stable bi-elliptic curve \bar{D} of genus 2 of compact type can be described as follows. Take elliptic curves D, D' and a subgroup $G \subset D[2] \times D'[2]$ such that G is isomorphic to $(\mathbb{Z}/2)^2$ and $G \cap (D \times \{0\}) = G \cap (\{0\} \times D') = \{0\}$. Then the double of the product polarisation on $D \times D'$ induces a principal polarisation Θ on $A = (D \times D')/G$. If $\bar{D} \subset A$ is a curve in the class of Θ , then \bar{D} is stable of genus 2 of compact type and the map $A \rightarrow D/D[2] \cong D$ restricts to a degree two map $\pi: \bar{D} \rightarrow D$.

If moreover A contains a 1-dimensional subgroup \bar{F} such that $\bar{D}\bar{F} = 3$, then the quotient map $A \rightarrow A/\bar{F} := E$ induces a tri-elliptic map $q: \bar{D} \rightarrow E$. To construct \bar{F} we construct an appropriate 1-dimensional subgroup $F \subset D \times D'$ resulting in a diagram

$$\begin{array}{ccc}
 F & \xrightarrow{\varphi \times \varphi'} & D \times D' \\
 \downarrow h_F & & \downarrow /G \\
 \bar{F} & \hookrightarrow & A = (D \times D')/G \xrightarrow{q} E = A/\bar{F} \\
 & \searrow m:1 & \downarrow \pi \\
 & & D
 \end{array}$$

where h_F is the quotient map induced by the action of G . From this situation we extract a numerical invariant of the bi-tri-elliptic configuration,

$$(3.10) \quad m = m(\bar{D}, \pi, q) := \deg(\bar{F} \rightarrow D) = \frac{4 \deg \varphi}{\deg h_F},$$

which we call the twisting number of (\bar{D}, π, q) . We now give two instances of this construction.

Example 3.11 — Let F be an elliptic curve and let $\varphi: F \rightarrow D$ and $\varphi': F \rightarrow D'$ isogenies such that $\ker \varphi \cap \ker \varphi' = \{0\}$, so that $\varphi \times \varphi': F \rightarrow D \times D'$ is injective. We abuse notation and denote again by F the image of $\varphi \times \varphi'$. Assume that $\deg \varphi + \deg \varphi' = 6$ and $\deg \varphi$ is odd. Set $G := F[2] \subset D \times D'$: for $i = 1, 2$ we have $G \cap (D \times \{0\}) = G \cap (\{0\} \times D') = \{0\}$ since $D \cdot F = \deg \varphi$ and $D' \cdot F = \deg \varphi'$ are odd. Let \bar{F} be the image of F in $A := (D \times D')/G$. If Θ is the principal polarisation of A , then we have $4\Theta\bar{F} = 2(D \times \{0\} + \{0\} \times D') \cdot F = 2(\deg \varphi' + \deg \varphi) = 12$, namely $\Theta\bar{F} = 3$.

The twisting number in this example is $m = \deg \varphi$.

Example 3.12 — We proceed as in Example 3.11, but here φ and φ' satisfy $\deg \varphi + \deg \varphi' = 3$. It is possible to find a subgroup $G \subset D[2] \times D'[2]$ of order 4 such that $G \cap (D \times \{0\}) = G \cap (\{0\} \times D') = \{0\}$ and $G \cap F$ has order 2. In fact, there are precisely four choices for such a subgroup. Again, we denote by \bar{F} the image of F in $(D \times D')/G$ and we compute $4\bar{F}\Theta = (2(D \times \{0\} + \{0\} \times D')) \cdot (2F) = 4(\deg \varphi' + \deg \varphi) = 12$, that is, $\bar{F}\Theta = 3$.

The twisting number of the resulting bi-tri-elliptic configuration is $m = 2 \deg \varphi$.

These two examples cover all possible cases.

Proposition 3.13 ([FPR16, Cor. 4.1]) — *Let (\bar{D}, π, q) be a bi-tri-elliptic configuration on a stable curve of genus 2. Then the twisting number m defined in (3.10) satisfies $1 \leq m \leq 5$ and there are the following possibilities:*

- (i) m is odd and the configuration arises as in Example 3.11 with $\deg \varphi = m$;
- (ii) m is even and the configuration arises as in Example 3.12 with $\deg \varphi = \frac{m}{2}$.

Remark 3.14 — We now consider the case that the bi-tri-elliptic curve in (3.6) is singular. It is shown in [FPR16, Sect. 4] that there is an elliptic curve D admitting an endomorphism $\psi: D \rightarrow D$ of degree 2 such that $\bar{D} = D \cup_0 D$ and the bi-tri-elliptic configuration is given by

$$\begin{array}{ccc} & D \cup_0 D & \\ \pi = \text{id} \cup \text{id} \swarrow & & \searrow q = \text{id} \cup \psi \\ D & & D \\ & 2:1 & 3:1 \end{array} .$$

In [FPR16, Sect. 4] it is also shown that there are exactly ten such configurations, two for each possible twisting number $1 \leq m \leq 5$.

3.D.3. *Classification of surfaces.* To exploit the connection between bi-tri-elliptic configurations and stable Godeaux surfaces we make the following definition.

Definition 3.15 — Let X be a Gorenstein stable Godeaux surface with normalisation the symmetric product of an elliptic curve and let (\bar{D}, π, q) be the associated bi-tri-elliptic configuration. We then say X is of type (E_m) , where m is the twisting number of (\bar{D}, π, q) defined in (3.10).

Then Propositions 3.8 and 3.13 immediately yield:

Proposition 3.16 — *Let X be a Gorenstein stable Godeaux surface with normalisation the symmetric product of an elliptic curve. Then X is of type (E_m) for some $1 \leq m \leq 5$.*

3.E. Calculating fundamental groups.

Proposition 3.17 — *Let X be a Gorenstein stable Godeaux surface with normalisation \mathbb{P}^2 . Then*

- (i) *If X is as in case (P_1) , including $X \cong X_{1,4}$, then $\pi_1(X) \cong \mathbb{Z}/4$.*
- (ii) *If X is as in case (P_2) , including $X \cong X_{1,1}$ or $X \cong X_{1,2}$, then $\pi_1(X) \cong \{1\}$.*
- (iii) *If X is as in case (P_3) , including $X \cong X_{1,3}$, then $\pi_1(X) \cong \mathbb{Z}/3$.*
- (iv) *If $X \cong X_{1,5}$ then $\pi_1(X) \cong \mathbb{Z}/5$.*

Proof. The proof relies on [FPR15a, Cor. 3.2 (ii)], which tells us that in the notation of Section 3.A we have

$$\pi_1(X) \cong \pi_1(\mathbb{P}^2) *_{\pi_1(\bar{D})} \pi_1(D) \cong \frac{\pi_1(D)}{R},$$

where $R = \langle\langle \pi_* \pi_1(\bar{D}) \rangle\rangle$ is the smallest normal subgroup of $\pi_1(D)$ generated by the image of $\pi_1(\bar{D})$. We make use of the explicit descriptions in Section 3.C and [FPR15a, Table 1].

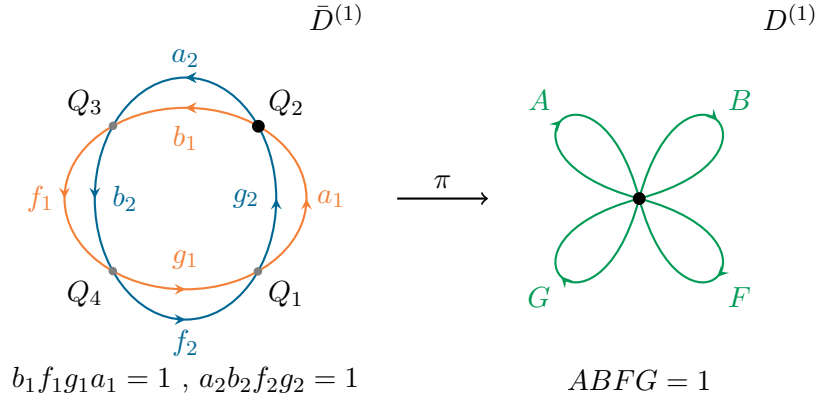
Note that topologically every irreducible component of \bar{D} is a 2-sphere; to do the computation we consider the following cell-decomposition: on every conic we arrange the four intersection points on a great circle, depicted as an ellipse, and consider these as the 0-skeleton and 1-skeleton. The sphere is formed by capping with two 2-cells.

On every line we choose the three intersection points as 0-cells, two intervals as 1-cells, and then glue in a 2-cell to make up the sphere. The order of the intersection points in the 1-skeleton will be chosen such that $\pi: \bar{D} \rightarrow D$ is easily visualised, that is, the involution is given by the obvious identification of oriented 1-skeletons.

In each case we will give the map between the 1-skeletons of $\bar{D}^{(1)}$ and $D^{(1)}$ of \bar{D} and D , and denote by \bullet a chosen base-point. We will name paths in the 1-skeleton of \bar{D} with lower case letters which will be mapped by π to paths in the 1-skeleton of D marked by the matching upper case letters. The indices will indicate the irreducible component the path belongs to. For each conic, the two 2-cells give one relation in the fundamental group, which we will indicate below the figures; the 2-cells of the lines do not affect the fundamental group.

Note that the shape of the 1-skeleton of the non-normal locus D does in some cases differ from the mnemonic representation in Figure 2.

general in (P_1) : The map on 1-skeletons and the extra relations are as follows:

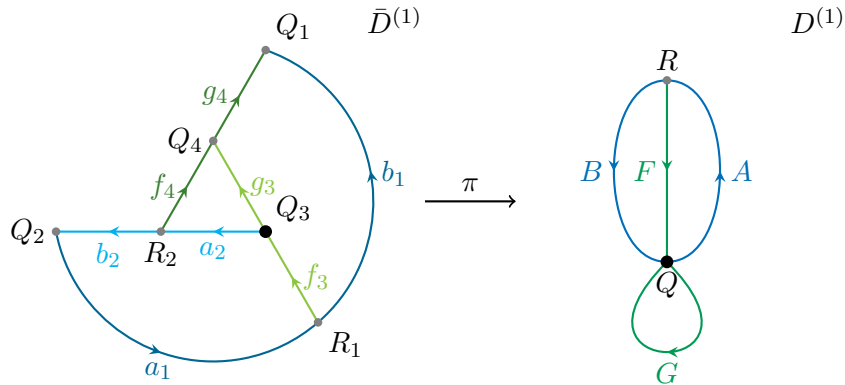


So the fundamental group of \bar{D} is the free group on generators $a_2 b_1^{-1}$, $a_1^{-1} g_2$, and $b_1 b_2 g_1 g_2$. Thus

$$\pi_1(X) = \langle A, B, G \mid AB^{-1}, A^{-1}G, B^2G^2 \rangle \cong \mathbb{Z}/4,$$

because the first two relations give $A = B = G$ in $\pi_1(X)$.

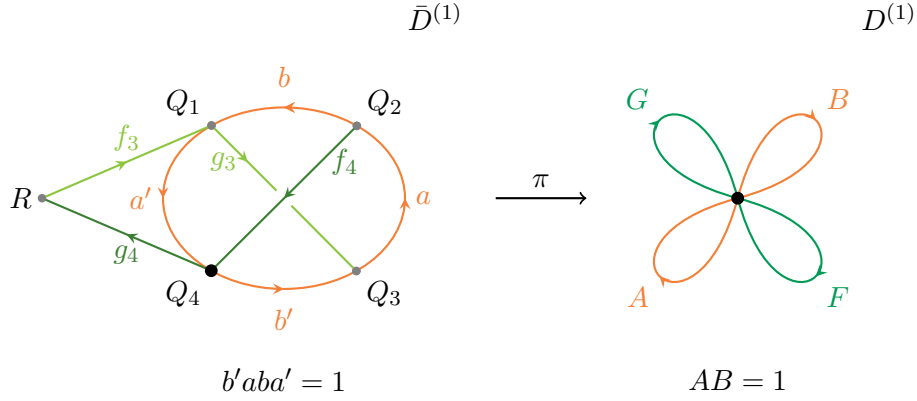
$X_{1.4}$: In this degenerate case of (P_1) the conic becomes two lines and we have four components $\bar{D}_1 = L_{12}$, $\bar{D}_3 = L_{34}$, $\bar{D}_2 = L_{23}$, $\bar{D}_4 = L_{14}$, where the involution is induced by the permutation (1234) . The newly developed nodes will be denoted by $R_1 = \bar{D}_1 \cap \bar{D}_3$ and $R_2 = \bar{D}_2 \cap \bar{D}_4$. Thus the map on 1-skeletons is



Again, $\pi_1(\bar{D})$ is free, generated by $a_2 f_4 g_3^{-1}$, $g_3 g_4 b_1^{-1} f_3$, and $a_2 b_2 a_1 f_3$, so that

$$\begin{aligned} \pi_1(X) &\cong \langle AB, AF, G \mid (AF)G^{-1}, G^2 B^{-1} F, (AB)(AF) \rangle \\ &\cong \langle AB, AF, G \mid (AF)G^{-1}, G^2 (AB)^{-1} (AF), (AB)(AF) \rangle \\ &\cong \langle G \mid G^4 \rangle \\ &\cong \mathbb{Z}/4. \end{aligned}$$

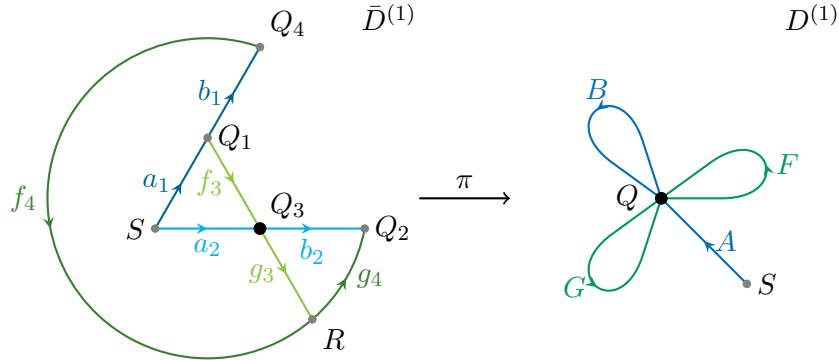
general in (P_2) : Recall that in this case the involution τ preserves the conic C . We can identify the conic with $\mathbb{C} \cup \{\infty\}$ in such a way that the intersection points are $\pm 1, \pm i$, which divide the unit circle in four arcs a, b, a', b' . In C/τ the unit circle is divided in two arcs A and B and clearly $BA = 1$ in $\pi_1(C/\tau)$. Adding in the two lines the map on 1-skeletons and the relations are as follows:



Again, $\pi_1(\bar{D})$ is free, generated by $g_4 f_3 a'$, $b' a f_4$, and $b' g_3^{-1} a'$, so that

$$\begin{aligned} \pi_1(X) &\cong \langle A, B, F, G \mid AB, GFA, BAF, BG^{-1}A \rangle \\ &\cong \langle A, F, G \mid GFA, F, A^{-1}G^{-1}A \rangle \\ &\cong \{1\}. \end{aligned}$$

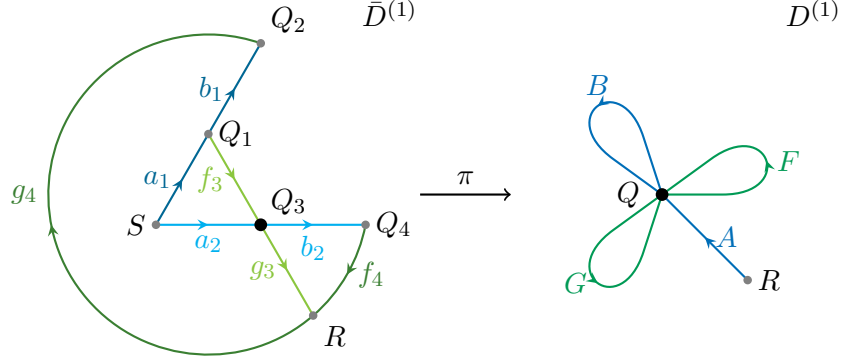
$X_{1,1}$: In this case the two components $\bar{D}_3 = L_{13}$, $\bar{D}_4 = L_{24}$ are exchanged by τ as described above and the two lines $\bar{D}_1 = L_{14}$, $\bar{D}_2 = L_{23}$ are exchanged via the involution induced by (13)(24). The new node $\bar{D}_1 \cap \bar{D}_2$ will be denoted by S . Then the map on 1-skeletons can be chosen to be



Again, $\pi_1(\bar{D})$ is free, generated by $a_2^{-1}a_1f_3$, $g_3g_4b_2^{-1}$, and $a_2^{-1}a_1b_1f_4g_3^{-1}$, so that

$$\begin{aligned}\pi_1(X) &\cong \langle B, F, G \mid F, G^2B^{-1}, BFG^{-1} \rangle \\ &\cong \langle B, G \mid G^2B^{-1}, BG^{-1} \rangle \\ &\cong \{1\}.\end{aligned}$$

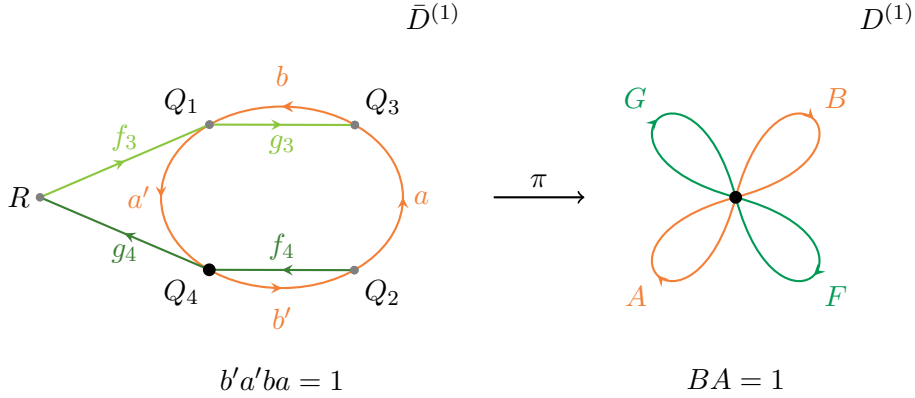
$X_{1,2}$: Again the two components $\bar{D}_3 = L_{13}$, $\bar{D}_4 = L_{24}$ are exchanged by τ as described above. The two lines $\bar{D}_1 = L_{12}$, $\bar{D}_2 = L_{34}$ are exchanged via the involution induced by (13)(24). The new node $\bar{D}_1 \cap \bar{D}_2$ will be denoted by S . Then the map on 1-skeletons can be chosen to be



Again, $\pi_1(\bar{D})$ is free, generated by $a_2^{-1}a_1f_3$, $g_3f_4^{-1}b_2^{-1}$, and $a_2^{-1}a_1b_1g_4^{-1}g_3^{-1}$, so that

$$\begin{aligned}\pi_1(X) &\cong \langle B, F, G \mid F, GF^{-1}B^{-1}, BG^{-2} \rangle \\ &\cong \langle B, G \mid G^{-1}B^{-1}, BG^{-2} \rangle \\ &\cong \{1\}.\end{aligned}$$

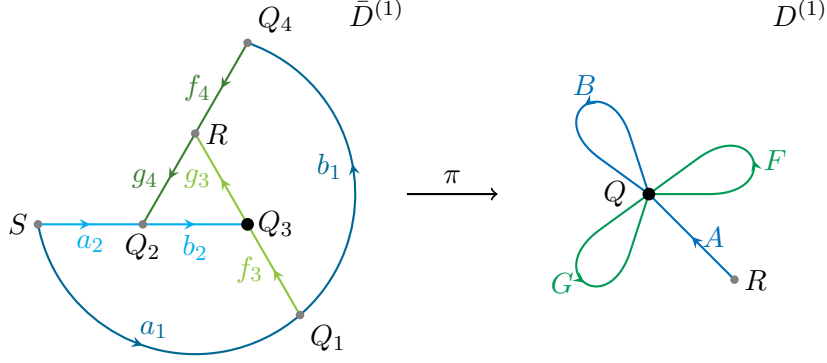
general in (P_3) : The map on 1-skeletons and the extra relations are as follows:



Again, $\pi_1(\bar{D})$ is free, generated by g_4f_3a' , $b'f_4$, and $b'ag_3^{-1}a'$, so that

$$\begin{aligned}\pi_1(X) &\cong \langle A, B, F, G \mid AB, GFA, BF, BAG^{-1}A \rangle \\ &\cong \langle A, F, G \mid GFA, A^{-1}F, G^{-1}A \rangle \\ &\cong \langle A \mid A^3 \rangle \cong \mathbb{Z}/3.\end{aligned}$$

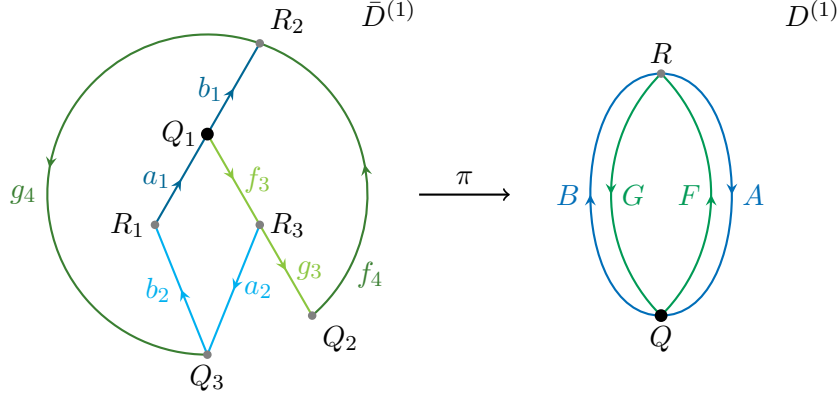
$X_{1,3}$: In this degenerate case of (P_3) the two components $\bar{D}_3 = L_{13}$, $\bar{D}_4 = L_{24}$ are exchanged by τ as described above and the two lines $\bar{D}_1 = L_{14}$, $\bar{D}_2 = L_{23}$ are exchanged via the involution induced by (12)(34). The new node $\bar{D}_1 \cap \bar{D}_2$ will be denoted by S . Then the map on 1-skeletons is



Again, $\pi_1(\bar{D})$ is free, generated by $g_3g_4b_2$, $f_3^{-1}b_1f_4g_3^{-1}$, and $f_3^{-1}a_1^{-1}a_2b_2$, so that

$$\begin{aligned} \pi_1(X) &\cong \langle B, F, G \mid G^2B, F^{-1}BFG^{-1}, F^{-1}B \rangle \\ &\cong \langle G \mid G^3 \rangle \\ &\cong \mathbb{Z}/3. \end{aligned}$$

$X_{1.5}$: This case does not come as the degeneration of the previous ones and we deviate slightly from the notation: the surface $X_{1.5}$ has two degenerate cusps at points Q and R and we mark the corresponding preimages with Q_i respectively R_i . The map on 1-skeletons can be given as



Again, $\pi_1(\bar{D})$ is free, generated by $b_1g_4b_2a_1$, $f_3g_3f_4b_1^{-1}$, and $f_3a_2b_2a_1$, so that

$$\begin{aligned} \pi_1(X) &\cong \langle BA, FA, FG \mid BGBA, FGF B^{-1}, FAB A \rangle \\ &\cong \langle AB, AF, GF \mid (BA)(FA)^{-1}(FG)(BA), (FG)(FA)(BA)^{-1}, (FA)(BA) \rangle \\ &\cong \langle BA, FA \mid (BA)^2(FG)(BA), (FG)(BA)^{-2} \rangle \\ &\cong \langle BA \mid (BA)^5 \rangle \\ &\cong \mathbb{Z}/5. \end{aligned}$$

Indeed, the universal cover of $X_{1.5}$ is a $\mathbb{Z}/5$ -invariant quintic in \mathbb{P}^3 , which is the union of 5 planes (compare [FR17, Rem. 9], Rem. 3.6 in the ArXiv version). \square

Proposition 3.18 — *Let X be a Gorenstein stable Godeaux surface.*

- (i) *If X is of type (E_m) ($m = 1, \dots, 5$) with irreducible polarisation then $\pi_1(X) \cong \mathbb{Z}/m$.*

(ii) If X is of type (E_m) ($m = 1, \dots, 5$) with reducible polarisation then X is simply connected.

Proof. (i) We compute the fundamental group case by case.

Consider the situation of Example 3.11 with \bar{D} smooth:

$$\begin{array}{ccc} F & \xrightarrow{\varphi \times \varphi'} & D \times D' \\ \downarrow & & \downarrow \\ \bar{F} = F/F[2] & \longrightarrow & \frac{D \times D'}{F[2]} = A \xrightarrow{q} A/\bar{F} \\ & & \downarrow \pi \\ & & D/D[2] \end{array}$$

Let $d = \deg \varphi$, $d' = \deg \varphi'$ and recall that $d + d' = 6$ and both numbers are odd. For all the elliptic curves we choose lattices in such a way that all morphisms in the diagram are induced by the identity on \mathbb{C}^2 . More specifically, we choose a non-real complex number τ such that $F = \mathbb{C}/\langle 2d, 2d'\tau \rangle$. In this way the integral first homology group of each curve in the above diagram can be read as a subgroup of \mathbb{C} and we have the following description of the induced maps in homology

$$\begin{aligned} H_1(F) &= \langle 2d, 2d'\tau \rangle \subset H_1(\bar{F}) = \langle d, d'\tau \rangle, \\ H_1(D) &= \langle 2, 2d'\tau \rangle \subset H_1(D/D[2]) = \langle 1, d'\tau \rangle, \\ H_1(D') &= \langle 2d, 2\tau \rangle. \end{aligned}$$

This is obvious if $\{d, d'\} = \{1, 5\}$; if $d = d' = 3$ then this is easy to achieve, since $F[3] = \ker \varphi \oplus \ker \varphi'$. With these choices we have

$$(3.19) \quad H_1(A) = \langle (2, 0), (2d'\tau, 0), (0, 2d), (0, 2\tau), (d, d), (d'\tau, d'\tau) \rangle \subset \mathbb{C}^2.$$

We denote by $a = 1$ and $b = d'\tau$ the generators of $H_1(D/D[2])$ and by α and β generators of $H_1(A/\bar{F})$ (to be specified below).

Recall from Proposition 3.8 that X is the stable surface associated to the triple $(\bar{X} = S^2(A/\bar{F}), \bar{D}, \sigma)$, where σ is the covering involution for the double cover $\pi: \bar{D} \rightarrow D/D[2]$. Thus by [FPR15a, Cor. 3.2 (iii)]. we have

$$(3.20) \quad \begin{aligned} \pi_1(X) &= H_1(D/D[2]) *_{H_1(\bar{D})} H_1(A/\bar{F}) \\ &= \langle a, b, \alpha, \beta \mid [a, b] = [\alpha, \beta] = 0, \pi_* \gamma_i = q_* \gamma_i \rangle, \end{aligned}$$

where the γ_i generate $H_1(\bar{D}) = H_1(A)$.

Case $d = 1$: In this case (3.19) becomes

$$H_1(A) = \langle (0, 2), (0, 2\tau), \underbrace{(1, 1), (5\tau, 5\tau)}_{=H_1(\bar{F})} \rangle$$

so that we can choose $\alpha = q_*(0, 2)$ and $\beta = q_*(0, 2\tau)$. Writing out the induced relations in (3.20) we get

$$\alpha = 0, \beta = 0, a = 0, b = 0,$$

and thus $\pi_1(X)$ is a trivial group.

Case $d = 3$: In this case (3.19) becomes

$$H_1(A) = \langle (2, 0), (0, 2\tau), \underbrace{(3, 3), (3\tau, 3\tau)}_{=H_1(\bar{F})} \rangle$$

so that we can choose $\alpha = q_*(2, 0)$ and $\beta = q_*(0, 2\tau)$. Writing out the induced relations in (3.20) we get

$$\alpha = 2a, \beta = 0, 3a = 0, b = 0,$$

and thus $\pi_1(X) \cong \mathbb{Z}/3\mathbb{Z}$.

Case $d = 5$: In this case (3.19) becomes

$$H_1(A) = \langle (2, 0), (2\tau, 0), \underbrace{(5, 5), (\tau, \tau)}_{=H_1(\bar{F})} \rangle$$

so that we can choose $\alpha = q_*(2, 0)$ and $\beta = q_*(2\tau, 0)$. Writing out the induced relations in (3.20) we get

$$\alpha = 2a, \beta = 2b, 5a = 0, b = 0,$$

and thus $\pi_1(X) \cong \mathbb{Z}/5\mathbb{Z}$.

We now consider the situation of Example 3.12 with irreducible polarisation:

$$\begin{array}{ccc} F & \xrightarrow{\varphi \times \varphi'} & D \times D' \\ \downarrow & & \downarrow \\ \bar{F} = F/F \cap G & \longrightarrow & \frac{D \times D'}{G} = A \xrightarrow{q} A/\bar{F} \\ & & \downarrow \pi \\ & & D/D[2] \end{array} .$$

Let $d = \deg \varphi$, $d' = \deg \varphi'$ and recall that $d + d' = 3$. The subgroup $G \cong (\mathbb{Z}/2\mathbb{Z})^2$ is chosen inside $D[2] \times D'[2]$ so that $G \cap (D \times \{0\}) = G \cap (\{0\} \times D') = \{0\}$ and $G \cap F = \langle \xi \rangle$ where ξ is a non-zero element in $F[2]$. If we denote by ζ the only non-zero element in $\ker \varphi + \ker \varphi' \subset F[2]$ (one of the maps has degree 1 and the other one degree 2), then by construction $F[2]$ is generated by ξ and ζ .

As in the first part of the proof, we choose a convenient representation of the homology lattices. We can identify $F = \mathbb{C}/\langle 4, 2\tau \rangle$ so that τ is in the upper half plane, ξ is the image of τ , and ζ is the image of 2. Then the other lattices are naturally described as follows:²

$$\begin{aligned} H_1(F) &= \langle 4, 2\tau \rangle \subset H_1(\bar{F}) = \langle 4, \tau \rangle, \\ H_1(D) &= \langle 2d', 2\tau \rangle \subset H_1(D/D[2]) = \langle d', \tau \rangle, \\ H_1(D') &= \langle 2d, 2\tau \rangle. \end{aligned}$$

A straightforward computation in $D[2] \times D'[2]$ shows that the only possibilities for G are

$$G_1 = \langle (\tau, \tau), (d', d) \rangle, G_2 = \langle (\tau, \tau), (d' + \tau, d) \rangle.$$

If we denote by $a = d'$ and $b = \tau$ the generators of $H_1(D/D[2])$ and with α and β generators for $H_1(A/\bar{F})$ we can compute the fundamental group as in (3.20).

²To check this, note that if $d' = 2$ then $H_1(F) \rightarrow H_1(D)$ should be the identity.

Case $d' = 1, G = G_1$: In this case

$$H_1(A) = \langle (2, 0), (2\tau, 0), (1, 2), \underbrace{(4, 4), (\tau, \tau)}_{=H_1(\bar{F})} \rangle,$$

$$2(1, 2) + (2, 0) \equiv 0 \pmod{H_1(\bar{F})}.$$

We can choose $\beta = q_*(2\tau, 0)$, $\alpha = q_*(1, 2)$ so that we get the relations (3.20)

$$\alpha = a, \beta = 2b, 4a = 0, b = 0$$

and thus $\pi_1(X) \cong \mathbb{Z}/4\mathbb{Z}$.

Case $d' = 1, G = G_2$: In this case

$$H_1(A) = \langle (2, 0), (0, 2\tau), (1 + \tau, 2), \underbrace{(4, 4), (\tau, \tau)}_{=H_1(\bar{F})} \rangle,$$

$$2(1 + \tau, 2) + (2, 0) + (0, 2\tau) \equiv 0 \pmod{H_1(\bar{F})}.$$

We can choose $\alpha = q_*(0, 2\tau)$, $\beta = q_*(1 + \tau, 2)$ so that the relations in (3.20) become

$$\alpha = 0, \beta = a + b, 4a = 0, b = 0$$

and thus $\pi_1(X) \cong \mathbb{Z}/4\mathbb{Z}$.

Case $d' = 2, G = G_1$: In this case

$$H_1(A) = \langle (0, 2), (0, 2\tau), (2, 1), \underbrace{(4, 4), (\tau, \tau)}_{=H_1(\bar{F})} \rangle,$$

$$2(2, 1) + (0, 2) \equiv 0 \pmod{H_1(\bar{F})}.$$

We can choose $\alpha = q_*(0, 2\tau)$ and $\beta = q_*(2, 1)$ so that the relations in (3.20) become

$$\alpha = 0, \beta = a, 2a = 0, b = 0$$

and thus $\pi_1(X) \cong \mathbb{Z}/2\mathbb{Z}$.

Case $d' = 2, G = G_2$: In this case

$$H_1(A) = \langle (0, 2), (0, 2\tau), (2 + \tau, 1), \underbrace{(4, 4), (\tau, \tau)}_{=H_1(\bar{F})} \rangle,$$

$$2(2 + \tau, 1) + (0, 2) + (0, 2\tau) \equiv 0 \pmod{H_1(\bar{F})}.$$

We can choose $\alpha = q_*(0, 2\tau)$ and $\beta = q_*(2 + \tau, 1)$, so that the relations in (3.20) become

$$\alpha = 0, \beta = a + b, 2a = 0, b = 0$$

and thus $\pi_1(X) \cong \mathbb{Z}/2\mathbb{Z}$.

(ii) Now assume that E is an elliptic curve and $\bar{D} \subset S^2E$ is a reducible bi-tri-elliptic curve of genus 2. Then as explained in Remark 3.14 we have $\bar{D} = C_1 + C_2$ where both $C_1 \cong C_2 \cong E$, C_1 is a section, and C_2 is a bisection of the Albanese map such that C_1 and C_2 meet in one point. Furthermore, the bi-elliptic map $\bar{D} \rightarrow D = C_1$ is induced by the identity on C_1 and restricts to an isomorphism $\varphi: C_2 \rightarrow C_1$.

Thus the subgroup $\pi_1(C_1) * \{1\} \subset \pi_1(\bar{D}) = \pi_1(C_1) * \pi_1(C_2)$ maps isomorphically both onto $\pi_1(\bar{X}) \cong \pi_1(E)$ and onto $\pi_1(D) = \pi_1(C_1)$. Hence

$$\begin{aligned} \pi_1(X) &\cong \pi(\bar{X}) *_{\pi_1(\bar{D})} \pi_1(D) \\ &\cong \pi_1(C_1) *_{(\pi_1(C_1) * \pi_1(C_2))} \pi_1(C_1) \\ &\cong \frac{\pi_1(C_1)}{\varphi_* \pi_1(C_2)} \cong \{1\}, \end{aligned}$$

because φ_* is an isomorphism. □

4. CONCLUSIONS

In this section we collect the results of the previous sections and in particular prove all claims contained in Table 1.

Theorem 4.1 — *Let X be a Gorenstein stable Godeaux surface with non-canonical singularities.*

- (i) *If X is normal then it is either of type (R) or (B_1) or (B_2) described in Section 2.A.*
- (ii) *If X is not normal, then X either has normalisation \mathbb{P}^2 and is as described in Proposition 3.4, or the normalisation of X is a symmetric product of elliptic curves and X is as in Proposition 3.16, or the normalisation of X is a del Pezzo surface of degree 1 and X is as described in Section 3.B.*

Proof. The first item is Proposition 2.1 with Proposition 2.4.

The second item follows from [FPR15b, Thm. 3.6] together with Propositions 3.4, 3.16, and 3.2. □

Corollary 4.2 — *Let X be a Gorenstein stable Godeaux surface. Then $\pi_1^{alg}(X)$ is cyclic of order at most 5. If X has non-canonical singularities or $|\pi_1^{alg}(X)| \geq 3$ then in addition $\pi_1(X) = \pi_1^{alg}(X)$.*

Proof. The case of canonical singularities is in [Miy76], and the equality between algebraic and topological fundamental group in the case of large torsion follows from the classification in [Rei78].

The cases with non-canonical singularities are listed in Theorem 4.1 and their fundamental groups have been computed in Theorem 2.5, Propositions 3.17 and 3.18 and [Rol16, Prop. 2.3]. Since in every case these are finite cyclic of order at most 5, they coincide with the algebraic fundamental groups. □

It is an interesting question, whether the examples that we construct are smoothable, that is, occur as degenerations of smooth Godeaux surfaces. As so far we can only attack this problem by an indirect method, i. e., by extending the classification of classical Godeaux surfaces with $|T(X)| \geq 3$ due to Reid [Rei78] to Gorenstein stable surfaces.

Theorem 4.3 (Theorem 1 in [FR17] (Thm. A in the ArXiv version)) — *Every Gorenstein stable Godeaux surface with $|T(X)| \geq 3$ is smoothable.*

On the other hand it was shown that the simply-connected examples of type (dP) are not smoothable [Rol16]. In other words, the moduli space of stable Godeaux

surfaces has at least one irreducible component which does not contain any smooth surface.

Unlike in the classical case, topological invariants do not distinguish between different connected components of the moduli space, since the topology of degenerations can differ drastically from the topology of a general fibre in the family. However, we can show the following.

Proposition 4.4 — *The compactification of the moduli space of Godeaux surfaces with torsion 0 or $\mathbb{Z}/2$ contains no normal Gorenstein surface with worse than canonical singularities.*

Note that the Gorenstein assumption is crucial as is shown by the examples in [LP07, Urz16]. Proposition 4.4 follows directly from Theorem 2.5 and the following general result.

Proposition 4.5 — *Let $\Delta \subset \mathbb{C}$ be the unit disk and let $\pi: \mathcal{X} \rightarrow \Delta$ be a proper flat family over Δ with reduced connected fibres.*

Denote by X_t the fibre over $t \in \Delta$; if $T(X_0)$ contains a cyclic subgroup of order d , then $T(X_t)$ also contains such a subgroup.

Proof. Let $Y_0 \rightarrow X_0$ be the étale cover associated to the cyclic subgroup of $T(X_0)$. It is well known that, possibly after shrinking Δ , the central fibre X_0 is a deformation retract of the total space \mathcal{X} [PS08, Rem. C.12], and therefore we may assume that the cover is induced by an étale cover $\mathcal{Y} \rightarrow \mathcal{X}$ of the total space. Our claim follows if the general fibre of the composition map $\mathcal{Y} \rightarrow \Delta$ is connected. This however follows by looking at the Stein factorisation because the central fibre Y_0 is reduced and connected by construction. \square

Remark 4.6 — Note however that the fundamental group need not be constant under degeneration. As an explicit example consider the following: consider a bi-tri-elliptic smoothing of a nodal bi-tri-elliptic curve. Then performing the construction of the surface of type (E_m) in families, we see that the one with reduced polarisation occurs as a degeneration of surfaces of type (E_m) with irreducible polarisation. By Proposition 3.18 the former is simply connected while the latter has cyclic fundamental group of order $m \leq 5$, so π_1 gets smaller in the special fibre.

On the other hand, since Godeaux surfaces with $|T(X)| \geq 3$ are smoothable by [FR17, Thm. 1], this construction implies that all surfaces of type (E_m) are smoothable if $m \geq 3$.

We do not dare speculate, whether it is at these points that the moduli space of stable Godeaux surfaces becomes connected.

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