CHARACTERISTIC IDEALS AND SELMER GROUPS

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Abstract. Let $A$ be an abelian variety defined over a global field $F$ of positive characteristic $p$ and let $F/F$ be a $\mathbb{Z}_p$-extension, unramified outside a finite set of places of $F$. Assuming that all ramified places are totally ramified, we define a pro-characteristic ideal associated to the Pontrjagin dual of the $p$-primary Selmer group of $A$. To do this we first show the relation between the characteristic ideals of duals of Selmer groups for a $\mathbb{Z}_d$-extension $F_d/F$ and for any $\mathbb{Z}_d^{-1}$-extension contained in $F_d$, and then use a limit process. Finally, we give an application to an Iwasawa Main Conjecture for the non-noetherian commutative Iwasawa algebra $\mathbb{Z}_p[[\text{Gal}(F/F)]]$ in the case $A$ is a constant abelian variety.

1. Introduction

Let $F$ be a global function field of characteristic $p$ and $F/F$ a $\mathbb{Z}_p$-extension unramified outside a finite set of places, whose Galois group we denote by $\Gamma$. We take an abelian variety $A$ defined over $F$ and let $S_A$ be a finite set of places of $F$ containing exactly the primes of bad reduction for $A$ and those which ramify in $F/F$. For any extension $v$ of some place of $F$ to the algebraic closure $\overline{F}$ and for any finite extension $E/F$, we denote by $E_v$ the completion of $E$ with respect to $v$ and, if $L/F$ is infinite, we put $L_v := \bigcup E_v$, where the union is taken over all finite subextensions of $L$. We define the $p$-part of the Selmer group of $A$ over $E$ as

$$Sel(E) := \text{Sel}_A(E)_p := \text{Ker} \left\{ H^1_{fl}(X_E, A[p\infty]) \longrightarrow \prod_v H^1_{fl}(X_{E_v}, A)[p\infty] \right\}$$

(where $H^1_{fl}$ denotes flat cohomology, $X_E := \text{Spec}(E)$ and the map is the product of the natural restrictions at all places $v$ of $E$). For infinite algebraic extensions we define the Selmer groups by taking direct limits on all the finite subextensions. For any algebraic extension $K/F$, let $S(K)$ denote the Pontrjagin dual of $\text{Sel}(K)$ (other Pontrjagin duals will be indicated by the symbol $\wedge$).

For any infinite $p$-adic Lie extension $L/F$, let $\Lambda(L) := \mathbb{Z}_p[[\text{Gal}(L/F)]]$ be the associated Iwasawa algebra: we recall that, if $\text{Gal}(L/F) \simeq \mathbb{Z}_d$, then $\Lambda(L) \simeq \mathbb{Z}_p[[t_1, \ldots, t_d]]$ is a Krull domain. It is well known that $S(L)$ is a $\Lambda(L)$-module and its structure has been described in several recent papers (see, e.g., [14] for $\text{Gal}(L/F) \simeq \mathbb{Z}_p^d$ and [5] for the non abelian case).

When $S(L)$ is a finitely generated module over a noetherian abelian Iwasawa algebra, it is possible to associate to $S(L)$ a characteristic ideal which is a key ingredient in Iwasawa Main Conjectures. We are interested in the definition of the analogue of a characteristic ideal in $\Lambda(F)$ for $S(F)$ (a similar result providing a pro-characteristic ideal for the Iwasawa module of class groups is described in [4]).

If $R$ is a noetherian Krull domain and $M$ a finitely generated torsion $R$-module, the structure theorem for $M$ provides an exact sequence

$$0 \longrightarrow P \longrightarrow M \longrightarrow \bigoplus_{i=1}^n R/p_i^{e_i} R \longrightarrow Q \longrightarrow 0$$

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where the $p_i$'s are height 1 prime ideals of $R$ and $P$ and $Q$ are pseudo-null $R$-modules (i.e., torsion modules with annihilator of height at least 2). With this sequence one defines the characteristic ideal of $M$ as

$$Ch_R(M) := \prod_{i=1}^{n} p_i^{e_i}$$

(if $M$ is not torsion, we put $Ch_R(M) = 0$, moreover note that $M$ is pseudo-null if and only if $Ch_R(M) = (1)$). In commutative Iwasawa theory characteristic ideals provide the algebraic counterpart for the $p$-adic $L$-functions associated to Iwasawa modules (such as duals of Selmer groups).

We fix a $\mathbb{Z}_p$-basis $\{\gamma_i\}_{i \in \mathbb{N}}$ for $\Gamma := \text{Gal}(\mathcal{F}/F)$ and, for any $d \geq 0$, we let $\mathcal{F}_d \subset \mathcal{F}$ be the fixed field of $\{\gamma_i\}_{i \geq d}$. Then we have $\Lambda(\mathcal{F}) = \lim_{\leftarrow} \Lambda(\mathcal{F}_d)$ and $S(\mathcal{F}) = \lim_{\leftarrow} S(\mathcal{F}_d)$. Note that the filtration $\{\mathcal{F}_d\}$ of $\mathcal{F}$ is uniquely determined once the $\gamma_i$ have been fixed, but we allow complete freedom in their initial choice. Put $t_i := \gamma_i - 1$: the subring $\mathbb{Z}_p[[t_1, \ldots, t_d]]$ of $\Lambda(\mathcal{F})$ is isomorphic to $\Lambda(\mathcal{F}_d)$ and, by a slight abuse of notation, the two shall be identified in this paper. In particular, for any $d \geq 1$ we have $\Lambda(\mathcal{F}_d) = \Lambda(\mathcal{F}_{d-1})[[t_d]]$. Let $\pi_{d-1}^d : \Lambda(\mathcal{F}_d) \to \Lambda(\mathcal{F}_{d-1})$ be the canonical projection, denote its kernel by $I_{d-1}^d = (t_d)$ and put $\Gamma_{d-1}^d := \text{Gal}(\mathcal{F}_d/\mathcal{F}_{d-1})$.

Our goal is to define an ideal attached to $S(\mathcal{F})$ in the non-noetherian Iwasawa algebra $\Lambda(\mathcal{F})$: we will do this via a limit of the characteristic ideals $Ch_{\Lambda(\mathcal{F}_d)}(S(\mathcal{F}_d))$. Thus we need to study the relation between $\pi_{d-1}^d(Ch_{\Lambda(\mathcal{F}_d)}(S(\mathcal{F}_d)))$ and $Ch_{\Lambda(\mathcal{F}_{d-1})}(S(\mathcal{F}_{d-1}))$. A general technique to deal with this type of descent and ensure that the limit does not depend on the filtration has been described in [4, Theorem 2.13]. That theorem is based on a generalization of some results of [11, Section 3] (which directly apply to our algebras $\Lambda(\mathcal{F}_e)$, even without the generalization to Krull domains provided in [4]) and can be applied to the $\Lambda(\mathcal{F})$-module $S(\mathcal{F})$. In our setting [4, Theorem 2.13] reads as follows

**Theorem 1.1.** If, for every $d \geq 1$,

1. the $t_d$-torsion submodule of $S(\mathcal{F}_d)$ is a pseudo-null $\Lambda(\mathcal{F}_{d-1})$-module, i.e.,

$$Ch_{\Lambda(\mathcal{F}_{d-1})}(S(\mathcal{F}_d)_{t_d}) = Ch_{\Lambda(\mathcal{F}_{d-1})}(S(\mathcal{F}_d)I_{d-1}^d) = (1);$$

2. $Ch_{\Lambda(\mathcal{F}_{d-1})}(S(\mathcal{F}_d)/t_d) = Ch_{\Lambda(\mathcal{F}_{d-1})}(S(\mathcal{F}_d)/I_{d-1}^d) \subseteq Ch_{\Lambda(\mathcal{F}_{d-1})}(S(\mathcal{F}_{d-1}))$,

then the ideals $Ch_{\Lambda(\mathcal{F}_d)}(S(\mathcal{F}_d))$ form a projective system (with respect to the maps $\pi_{d-1}^d$). In Section 2 we show that if $S(\mathcal{F}_e)$ is $\Lambda(\mathcal{F}_e)$-torsion, then $S(\mathcal{F}_d)$ is $\Lambda(\mathcal{F}_d)$-torsion for all $d \geq e$ and use [4, Proposition 2.10] to provide a general relation

$$Ch_{\Lambda(\mathcal{F}_{d-1})}(S(\mathcal{F}_d)I_{d-1}^d) \cdot \pi_{d-1}^d(Ch_{\Lambda(\mathcal{F}_d)}(S(\mathcal{F}_d))) = Ch_{\Lambda(\mathcal{F}_{d-1})}(S(\mathcal{F}_{d-1}) \cdot J_d)$$

(see (2.9) where the extra factor $J_d$ is more explicit). Then we move to the totally ramified setting, i.e., extensions in which all ramified primes are assumed to be totally ramified (an example are the extensions obtained from $F$ by adding the $\mathfrak{a}^n$-torsion points of a normalized rank 1 Drinfeld module over $F$). In this setting, using some techniques and results of K.-S. Tan ([15]), we check the hypotheses of Theorem 1.1 using equation (1.2), and obtain (see Corollary 3.8 and Definition 3.9)

**Theorem 1.2.** Assume all ramified primes in $\mathcal{F}/F$ are totally ramified. Then, for $d \geq 0$,

$$\pi_{d-1}^d(Ch_{\Lambda(\mathcal{F}_d)}(S(\mathcal{F}_d))) = Ch_{\Lambda(\mathcal{F}_{d-1})}(S(\mathcal{F}_{d-1}))$$

and the pro-characteristic ideal

$$\bar{Ch}_{\Lambda(\mathcal{F})}(S(\mathcal{F})) := \lim_{\leftarrow} Ch_{\Lambda(\mathcal{F}_d)}(S(\mathcal{F}_d)) \subseteq \Lambda(\mathcal{F})$$

is well defined.
As an application, we use a deep result of Lai - Longhi - Tan - Trihan [9] to prove an Iwasawa Main Conjecture for constant abelian varieties in our non-Noetherian setting (see Theorem 3.10).

2. General $\mathbb{Z}_p$-descent for Selmer groups

To be able to define characteristic ideals we need the following

**Theorem 2.1. (Tan)** Assume that $A$ has good ordinary or split multiplicative reduction at all ramified places of the finite set $S_A$. Then, for any $d$ and any $\mathbb{Z}_p^d$-extension $\mathcal{L}/F$ contained in $\mathcal{F}$, the group $S(\mathcal{L})$ is a finitely generated $\Lambda(\mathcal{L})$-module.

**Proof.** In this form the theorem is due to Tan ([14, Theorem 5]). See also [3, Section 2] and the references there. □

If there is a place $v$ ramified in $\mathcal{L}/F$ and of supersingular reduction for $A$, then the module $S(\mathcal{L})$ is not finitely generated over $\Lambda(\mathcal{L})$ by [15, Proposition 1.1 and Theorem 3.10]. In order to obtain a nontrivial relation between the characteristic ideals, we need no ramified supersingular primes and something more than just Theorem 2.1, so we make the following

**Assumptions 2.2.**
1. All places ramified in $\mathcal{F}/F$ are of ordinary reduction.
2. There exists an $\epsilon > 0$ such that $S(\mathcal{F}_\epsilon)$ is a torsion $\Lambda(\mathcal{F}_\epsilon)$-module.

**Remarks 2.3.**
1. Hypothesis 2 is satisfied in many cases: for example when $\mathcal{F}_\epsilon$ contains the arithmetic $\mathbb{Z}_p$-extension of $F$ (proof in [15, Theorem 2], extending [12, Theorem 1.7]) or when $\text{Sel}(F)$ is finite and $A$ has good ordinary reduction at all places which ramify in $\mathcal{F}_\epsilon/F$ (easy consequence of [14, Theorem 4]).
2. Our goal is an equation relating $\pi^{d-1}_d(\text{Ch}_\Lambda(\mathcal{F}_d)S(\mathcal{F}_d))$ and the characteristic ideal of $S(\mathcal{F}_{d-1})$. If the above assumption 2 is not satisfied for any $\epsilon$, then all characteristic ideals are 0 and there is nothing to prove.

In this section we also assume that none of the ramified prime has trivial decomposition in $\text{Gal}(\mathcal{F}_1/F)$. In Section 3 we shall work in extensions in which ramified places are totally ramified, so this assumption will be automatically verified. Anyway this is not restrictive in general because of the following

**Lemma 2.4.** If $d \geq 2$, one can always find a $\mathbb{Z}_p$-subextension $\mathcal{F}_1/F$ of $\mathcal{F}_d/F$ in which none of the ramified places splits completely.

**Proof.** See [4, Lemma 3.1] □

Consider the diagram

\begin{equation}
\begin{array}{ccc}
\text{Sel}(\mathcal{F}_{d-1}) & \hookrightarrow & H^1_{fl}(\mathcal{X}_{d-1}, A[p^\infty]) \\
\downarrow_{\pi^{d-1}_d} & & \downarrow_{\psi^{d-1}_d} \\
\text{Sel}(\mathcal{F}_d)^{\Gamma_{d-1}} & \hookrightarrow & H^1_{fl}(\mathcal{X}_d, A[p^\infty])^{\Gamma_{d-1}} \\
\downarrow_{\phi^{d-1}_d} & & \downarrow_{\phi^{d-1}_d} \\
\end{array}
\end{equation}

where $\mathcal{X}_d := Spec(\mathcal{F}_d)$, the vertical maps are induced by (global) restrictions and $\mathcal{G}(\mathcal{X}_d)$ is the image of the product of the (local) restriction maps

$$H^1_{fl}(\mathcal{X}_d, A[p^\infty]) \longrightarrow \prod_w H^1_{fl}(\mathcal{X}_{d,w}, A)[p^\infty],$$

with $w$ running over all places of $\mathcal{F}_d$ where $\mathcal{X}_{d,w} := Spec(\mathcal{F}_{d,w})$ with $\mathcal{F}_{d,w}$ the completion of $\mathcal{F}_d$ at $w$. 
Lemma 2.5. Assume that no ramified place is totally split in $\mathcal{F}_1/F$. For any $d \geq 2$, the Pontrjagin dual of $\ker c_{d-1}^d$ is a finitely generated torsion $\Lambda(\mathcal{F}_{d-1})$-module.

Proof. For any place $v$ of $F$ we fix an extension to $\mathcal{F}$, which by a slight abuse of notation we still denote by $v$, so that the set of places of $\mathcal{F}_d$ above $v$ will be the Galois orbit $\text{Gal}(\mathcal{F}_d/F) \cdot v$. For any field $L$ let $\mathcal{P}_L$ be the set of places of $L$. We have an obvious injection

$$\ker c_{d-1}^d \hookrightarrow \prod_{u \in \mathcal{P}_{\mathcal{F}_{d-1}}} \ker \left\{ H^1_I(\chi_{d-1,u}, A)[p^\infty] \longrightarrow \prod_{w \mid u} H^1_I(\chi_{d,w}, A)[p^\infty] \right\}$$

(the map is the product of the natural restrictions $r_w$). By the Hochschild-Serre spectral sequence, we get

$$\ker r_w \simeq H^1(\Gamma_{d-1,w}, A(\mathcal{F}_{d,w}))[p^\infty]$$

where $\Gamma_{d-1,w}$ is the decomposition group of $w$ in $\Gamma_{d-1}^d$. Those kernels really depend only on the place $u$ of $\mathcal{F}_{d-1}$ lying below $w$ (for any $w_1, w_2$ dividing $u$ we obviously have $\ker r_{w_1} \simeq \ker r_{w_2}$).

Hence for any $v$ of $F$ and any $u \in \mathcal{P}_{\mathcal{F}_{d-1}}$ dividing it, we fix a $w(u)$ of $\mathcal{F}_d$ over $u$ and define

$$\mathcal{H}_v(\mathcal{F}_d) := \prod_{u \in \mathcal{P}_{\mathcal{F}_{d-1}}/v} \ker \left\{ H^1(\Gamma_{d-1,u(w)}, A(\mathcal{F}_{d,u(w)}))[p^\infty] \right\}.$$  

Equation (2.2) now reads as

$$\ker c_{d-1}^d \hookrightarrow \bigoplus_{v \in \mathcal{P}_{\mathcal{F}}} \mathcal{H}_v(\mathcal{F}_d).$$

Obviously $\mathcal{H}_v(\mathcal{F}_d) = 0$ for all primes which totally split in $\mathcal{F}_d/F_{d-1}$ and, from now on, we only consider places such that $\Gamma_{d-1,w(u)}^d \neq 0$.

Let $\Lambda(\mathcal{F}_{1,v}) := \mathbb{Z}_p[[\text{Gal}(\mathcal{F}_{1,v}/F_v)]]$ be the Iwasawa algebra associated to the decomposition group of $v$ in $\text{Gal}(\mathcal{F}_1/F)$ and note that each $\ker r_w$ is a $\Lambda(\mathcal{F}_{d-1,v})$-module. Moreover, we get an action of $\text{Gal}(\mathcal{F}_{d-1}/F)$ on $\mathcal{H}_v(\mathcal{F}_d)$ by permutation of the primes $u \in \mathcal{P}_{\mathcal{F}_{d-1}}$, $v$ and an isomorphism

$$\mathcal{H}_v(\mathcal{F}_d) \simeq \Lambda(\mathcal{F}_{d-1}) \otimes_{\Lambda(\mathcal{F}_{d-1,v})} H^1(\Gamma_{d-1,u(w)}^d, A(\mathcal{F}_{d,u(w)}))[p^\infty]$$

(see also [15, Lemma 3.2], note that $H^1(\Gamma_{d-1,u(w)}^d, A(\mathcal{F}_{d,u(w)}))[p^\infty]$ is finitely generated over $\Lambda(\mathcal{F}_{d-1,v})$).

First assume that the place $v$ is unramified in $\mathcal{F}_d/F$ (hence inert in $\mathcal{F}_d/F_{d-1}$). Then $\mathcal{F}_{d-1,v} = F_v \neq \mathcal{F}_{d,v}$ and one has, by $[10, \text{Proposition} \ 1.3.8]$,

$$H^1(\Gamma_{d-1,u(w)}^d, A(\mathcal{F}_{d,u(w)})) \simeq H^1(\Gamma_{d-1,u(w)}^d, \pi_0(\mathcal{A}_{0,v})), $$

where $\mathcal{A}_{0,v}$ is the closed fiber of the Néron model of $A$ over $F_v$ and $\pi_0(\mathcal{A}_{0,v})$ is its set of connected components. It follows that $H^1(\Gamma_{d-1,u(w)}^d, A(\mathcal{F}_{d,u(w)}))[p^\infty]$ is trivial when $v$ does not lie above $S_A$ and that it is finite of order bounded by (the $p$-part of) $|\pi_0(\mathcal{A}_{0,v})|$ for the unramified places of bad reduction. Hence (2.4) reduces to

$$\ker c_{d-1}^d \hookrightarrow \bigoplus_{v \in S'_A(d)} \mathcal{H}_v(\mathcal{F}_d)$$

(where $S'_A(d)$ is the set of primes in $S_A$ which are not totally split in $\mathcal{F}_d/F_{d-1}$) and, by (2.5), $\mathcal{H}_v(\mathcal{F}_d)^p$ is a finitely generated torsion $\Lambda(\mathcal{F}_{d-1})$-module for unramified $v$.

For the ramified case the exact sequence

$$A(\mathcal{F}_{d,u(w)}) \xrightarrow{p} A(\mathcal{F}_{d,u(w)}) \xrightarrow{p} pA(\mathcal{F}_{d,u(w)})$$
yields a surjection
\[ H^1(\Gamma_{d-1,w(u)}^d, A(\mathcal{F}_{d,w(u)})[p]) \longrightarrow H^1(\Gamma_{d-1,w(u)}^d, A(\mathcal{F}_{d,w(u)})[p]) \, . \]
The first module is obviously finite, so \( H^1(\Gamma_{d-1,w(u)}^d, A(\mathcal{F}_{d,w(u)})[p]) \) is finite as well: this implies that \( H^1(\Gamma_{d-1,w(u)}^d, A(\mathcal{F}_{d,w(u)})[p^\infty])^\vee \) has finite \( \mathbb{Z}_p \)-rank. As a finitely generated \( \mathbb{Z}_p \)-module, \( H^1(\Gamma_{d-1,w(u)}^d, A(\mathcal{F}_{d,w(u)})[p^\infty])^\vee \) must be \( \mathbb{Z}_p[[\Gamma_{d-1,w}]] \)-torsion for any \( d \geq 2 \) (because of our choice of \( \mathcal{F}_1/F \)) and (2.5) shows once again that \( \mathcal{H}_v(\mathcal{F}_d)^\vee \) is finitely generated and torsion over \( \Lambda(\mathcal{F}_{d-1}) \). \( \square \)

**Remark 2.6.** One can go deeper in the details and compute those kernels according to the reduction of \( A \) at \( v \) and the behavior of \( v \) in \( \mathcal{F}_d/F \). We will do this in Section 3 but only for the particular case of a totally ramified extension (with the statement of a Main Conjecture as a final goal). See [15] for a more general analysis.

The following proposition provides a crucial step towards equation (1.2) (in particular it also takes care of hypothesis 2 of Theorem 1.1).

**Proposition 2.7.** Assume that no ramified place is totally split in \( \mathcal{F}_1/F \). Let \( e \) be as in Assumption 2.2.2. For any \( d > e \), the module \( \mathcal{S}(\mathcal{F}_d)/I_{d-1}^d \) is a finitely generated torsion \( \Lambda(\mathcal{F}_{d-1}) \)-module and \( \mathcal{S}(\mathcal{F}_d) \) is a finitely generated torsion \( \Lambda(\mathcal{F}_d) \)-module. Moreover, if \( d > \max\{2, e\} \),

\[ Ch_{A(\mathcal{F}_{d-1})}(\mathcal{S}(\mathcal{F}_d)/I_{d-1}^d) = Ch_{A(\mathcal{F}_{d-1})}(\mathcal{S}(\mathcal{F}_{d-1})) \cdot Ch_{A(\mathcal{F}_{d-1})}((\text{Coker } a_{e}^{d-1})^\vee) \, . \]

**Proof.** It suffices to prove the first statement for \( d = e + 1 \), then a standard argument (detailed, e.g., in [8, page 207]) shows that \( \mathcal{S}(\mathcal{F}_{e+1}) \) is \( \Lambda(\mathcal{F}_{e+1}) \)-torsion and we can iterate the process. From diagram (2.1) one gets a sequence

\[ (\text{Coker } a_{e}^{e+1})^\vee \longrightarrow (\text{Sel}(\mathcal{F}_{e+1})^\vee) \longrightarrow \mathcal{S}(\mathcal{F}_e) \longrightarrow (\text{Ker } a_{e}^{e+1})^\vee \, . \]

By the Hochschild-Serre spectral sequence, it follows

\[ \text{Coker } b_{e}^{e+1} \overset{\sim}{\longrightarrow} H^2(\Gamma_{e}^{e+1}, A[p^\infty](\mathcal{F}_{e+1})) = 0 \]

(because \( \Gamma_{e}^{e+1} \) has \( p \)-cohomological dimension 1). Therefore there is a surjective map

\[ \text{Ker } c_{e}^{e+1} \longrightarrow \text{Coker } a_{e}^{e+1} \]

and, by Lemma 2.5, \( (\text{Coker } a_{e}^{e+1})^\vee \) is \( \Lambda(\mathcal{F}_e) \)-torsion. Hence Assumption 2.2.2 and sequence (2.7) yield that

\[ (\text{Sel}(\mathcal{F}_{e+1})^\vee) \simeq \mathcal{S}(\mathcal{F}_{e+1})/I_{e+1} \]

is \( \Lambda(\mathcal{F}_e) \)-torsion. To conclude note that (for any \( d \)) the duals of

\[ \text{Ker } a_{d-1}^{d} \longrightarrow \text{Ker } b_{d-1}^{d} \simeq H^1(\Gamma_{d-1}^d, A[p^\infty](\mathcal{F}_d)) \simeq A[p^\infty](\mathcal{F}_d)/I_{d-1}^d \]

are finitely generated \( \mathbb{Z}_p \)-modules (hence pseudo-null over \( \Lambda(\mathcal{F}_{d-1}) \) for any \( d \geq 3 \)). Taking characteristic ideals in the sequence (2.7), for large enough \( d \), one finds

\[ Ch_{A(\mathcal{F}_{d-1})}(\mathcal{S}(\mathcal{F}_d)/I_{d-1}^d) = Ch_{A(\mathcal{F}_{d-1})}(\mathcal{S}(\mathcal{F}_{d-1})) \cdot Ch_{A(\mathcal{F}_{d-1})}((\text{Coker } a_{d-1}^{d})^\vee) \, . \]

\( \square \)

**Remark 2.8.** In [12, Theorem 1.7], the authors prove that \( \mathcal{S}(F^{(p)}) \) is a finitely generated torsion \( \mathbb{Z}_p[[\text{Gal}(F^{(p)}/F)]] \)-module (where \( F^{(p)} \) is the arithmetic \( \mathbb{Z}_p \)-extension of \( F \)). The first part of the proof above provides a more direct approach to the generalization of this result given in [15, Theorem 2].
Whenever $\mathcal{S}(\mathcal{F}_d)$ is a finitely generated torsion $\Lambda(\mathcal{F}_d)$-module, [4, Proposition 2.10] yields
\begin{equation}
(2.8) \quad Ch_{\Lambda(\mathcal{F}_{d-1})}(\mathcal{S}(\mathcal{F}_d))^{\Gamma_{d-1}} \cdot \pi_{d-1}^{d}(Ch_{\Lambda(\mathcal{F}_d)}(\mathcal{S}(\mathcal{F}_d))) = Ch_{\Lambda(\mathcal{F}_{d-1})}(\mathcal{S}(\mathcal{F}_d)/I_{d-1})^d.
\end{equation}
If $d > \max\{2, e\}$, equation (2.8) turns into
\begin{equation}
(2.9) \quad Ch_{\Lambda(\mathcal{F}_{d-1})}(\mathcal{S}(\mathcal{F}_d))^{\Gamma_{d-1}} \cdot \pi_{d-1}^{d}(Ch_{\Lambda(\mathcal{F}_d)}(\mathcal{S}(\mathcal{F}_d))) = Ch_{\Lambda(\mathcal{F}_{d-1})}(\mathcal{S}(\mathcal{F}_d)) \cdot J_d,
\end{equation}
where $J_d := Ch_{\Lambda(\mathcal{F}_{d-1})}(((\text{Coker} a_{d-1}^d)^{\vee})$.
Therefore, whenever we can prove that $\mathcal{S}(\mathcal{F}_d)^{\Gamma_{d-1}}$ is a pseudo-null $\Lambda(\mathcal{F}_{d-1})$-module (i.e., hypothesis 1 of Theorem 1.1), we immediately get
\begin{equation}
(2.10) \quad \pi_{d-1}^{d}(Ch_{\Lambda(\mathcal{F}_d)}(\mathcal{S}(\mathcal{F}_d))) \subseteq Ch_{\Lambda(\mathcal{F}_{d-1})}(\mathcal{S}(\mathcal{F}_d))\end{equation}
and Theorem 1.1 will provide the definition of the pro-characteristic ideal for $\mathcal{S}(\mathcal{F})$ in $\Lambda(\mathcal{F})$ we were looking for.

3. $\mathbb{Z}_p$-descent for totally ramified extensions

The main examples we have in mind are extensions satisfying the following

Assumption 3.1. The (finitely many) ramified places of $\mathcal{F}/F$ are totally ramified.

In what follows an extension satisfying this assumption will be called a totally ramified extension. A prototypical example is the $a$-cyclotomic extension of $\mathbb{F}_q(T)$ generated by the $a$-torsion of the Carlitz module (a an ideal of $\mathbb{F}_q[T]$, see, e.g., [13, Chapter 12]). As usual in Iwasawa theory over number fields, most of the proofs will work (or can be adapted) simply assuming that ramified primes are totally ramified in $\mathcal{F}/F$ theory. Setting, one needs some extra hypothesis on the behaviour of these places in $\mathcal{F}/F$ for some $e \geq 0$, but, in the function field setting, one would need some extra hypothesis on the behaviour of these places in $\mathcal{F}_e/F$ (as we have seen with Lemma 2.4, note that in totally ramified extensions any $\mathbb{Z}_p$-subextension can play the role of $\mathcal{F}_1$).

A relevant example for the last case is the composition of a $a$-cyclotomic extension and of the arithmetic $\mathbb{Z}_p$-extension of $\mathbb{F}_q(T)$ (with the second one playing the role of $\mathcal{F}_1$). Note that Assumption 2.2.2 is verified in this case with $e = 1$, thanks to [12, Theorem 1.7], hence our next results hold for all these extensions as well.

Let $v \in S_A$ be unramified in $\mathcal{F}/F$, then it is either totally split or it is inert in just one $\mathbb{Z}_p$-extension $\mathcal{F}(d/v)$ and totally split in all the others. Since $|S_A|$ is finite we can fix an index $d_0$ such that all unramified places of $S_A$ are totally split in $\mathcal{F}/\mathcal{F}_{d_0}$.

Theorem 3.2. Assume $\mathcal{F}/F$ is a totally ramified extension, then, for any $d > \max\{d_0, 2\}$, we have
\[ Ch_{\Lambda(\mathcal{F}_{d-1})}((\text{Coker} a_{d-1}^d)^{\vee}) = (1). \]

Proof. The proof of Proposition 2.7 shows that the $\Lambda(\mathcal{F}_{d-1})$-modules $(\text{Coker} a_{d-1}^d)^{\vee}$ and $(\text{Ker} c_{d-1}^d)^{\vee}$ are pseudo-isomorphic for $d \geq 3$. Moreover, by the proof of Lemma 2.5 (recall, in particular, equation (2.6)), we know that $(\text{Ker} c_{d-1}^d)^{\vee}$ is a quotient of $\bigoplus_{v \in S_A(d)} \mathcal{H}_v(\mathcal{F}_d)^{\vee}$. Hence we only consider the contributions of the places of $S_A$ which are not totally split in $\mathcal{F}/F$. By equation (2.5), we have (for a fixed $v$ dividing $d$)
\begin{equation}
(3.1) \quad Ch_{\Lambda(\mathcal{F}_{d-1})}(\mathcal{H}_v(\mathcal{F}_d)^{\vee}) = \Lambda(\mathcal{F}_{d-1}) \otimes_{\Lambda(\mathcal{F}_{d-1}, v)} Ch_{\Lambda(\mathcal{F}_{d-1}, v)}(H^1(\Gamma_{d-1, v}, \Lambda(\mathcal{F}_{d, v}))[p^{\infty}]^{\vee}).
\end{equation}
We also saw that, for a ramified prime $v$, $\mathcal{H}_v(\mathcal{F}_d)^{\vee}$ (which is $H^1(\Gamma_{d-1, v}, \Lambda(\mathcal{F}_{d, v}))[p^{\infty}]^{\vee}$, because $v$ is totally ramified) is finitely generated over $\mathbb{Z}_p$, hence pseudo-null over $\Lambda(\mathcal{F}_{d-1, v}) = \Lambda(\mathcal{F}_{d-1})$ for $d \geq 3$. 

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We are left with the unramified (not totally split) primes in $S_A$. Assume $v$ is inert in an extension $\mathcal{F}_r/\mathcal{F}_{r-1}$ ($r \leq d_0$ by definition), then
\[ \Lambda(\mathcal{F}_{r-1,v}) \simeq \mathbb{Z}_p \quad \text{and} \quad \Lambda(\mathcal{F}_{d,v}) \simeq \mathbb{Z}_p[t_r] \] for any $d \geq r$.

Since (again by Lemma 2.5) $H^1(\Gamma_{r-1,w}, A(\mathcal{F}_{r,w}))[p^{\infty}]$ is finite and $H^1(\Gamma_{d-1,w}, A(\mathcal{F}_{d,w}))[p^{\infty}] = 0$ for any $d \geq r$, we have
\[ Ch_{\Lambda(\mathcal{F}_{r-1,v})}(H^1(\Gamma_{r-1,w}, A(\mathcal{F}_{r,w}))[p^{\infty}]) = (p^{\nu(v)}) \]
for some $\nu(v)$ depending on $|\pi_0(\mathcal{A}_{0,v})|$, and
\[ Ch_{\Lambda(\mathcal{F}_{d-1,v})}(H^1(\Gamma_{d-1,w}, A(\mathcal{F}_{d,w}))[p^{\infty}]) = (1) \] for any $d \geq d_0 + 1 \geq r + 1$.

These local informations and (3.1) yield the theorem. \qed

Now we deal with the other extra term of equation (2.9), i.e., $Ch_{\Lambda(\mathcal{F}_{d-1})}(S(\mathcal{F}_d)^{\Gamma_d}_{d-1})$. Note first that, taking duals
\[ (S(\mathcal{F}_d)^{\Gamma_d}_{d-1})^\vee \simeq S(\mathcal{F}_d)^\vee / (\gamma_d - 1) = Sel(\mathcal{F}_d) / (\gamma_d - 1), \]
so we work on the last module.

From now on we put $\gamma := \gamma_d$ and we shall need the following two lemmas: the first is [15, Proposition 4.4] (we provide the proof for completeness), while the second generalizes [15, Proposition 4.2].

**Lemma 3.3.** We have
\[ H^1_f(X_d, A[p^{\infty}]) = (\gamma - 1)H^1_f(X_d, A[p^{\infty}]). \]

**Proof.** Since
\[ H^1_f(X_d, A[p^{\infty}]) = \lim_{K \subset \mathcal{F}_d, [K:F] < \infty} \lim_{m \to \infty} H^1_f(X_K, A[p^m]), \]
an element $\alpha \in H^1_f(X_d, A[p^{\infty}])$ belongs to some $H^1_f(X_K, A[p^m])$. Now let $\gamma^{p^t(K)}$ be the largest power of $\gamma$ which acts trivially on $K$, and define a $\mathbb{Z}_p$-extension $K_\infty$ with $Gal(K_\infty/K) = \langle \gamma^{p^t(K)} \rangle$ and layers $K_n$. Take $t \geq m$, consider the restrictions
\[ H^1_f(X_K, A[p^m]) \to H^1_f(X_{K_t}, A[p^m]) \to H^1_f(X_{K_\infty}, A[p^m]) \]
and denote by $x_t$ the image of $x$. Now $x_t$ is fixed by $Gal(K_t/K)$ and $p^m x_t = 0$, so $x_t$ is in the kernel of the norm $N_{K_t}^{K_\infty}$, i.e., $x_t$ belongs to the (Galois) cohomology group
\[ H^1(K_t/K, H^1_f(X_{K_\infty}, A[p^m]) \to H^1(K_\infty/K, H^1_f(X_{K_\infty}, A[p^m]) \to H^1(K_\infty/K, A[p^m]). \]

Let $Ker^2_m$ be the kernel of the restriction map $H^2_f(X_K, A[p^m]) \to H^2_f(X_{K_\infty}, A[p^m])$, then, from the Hochschild-Serre spectral sequence, we have
\[ (3.2) \quad Ker^2_m \to H^1(K_\infty/K, H^1_f(X_{K_\infty}, A[p^m]) \to H^3(K_\infty/K, A[K_\infty)[p^m]) = 0 \]
(because the $p$-cohomological dimension of $\mathbb{Z}_p$ is 1). To get rid of $Ker^2_m$ note that, by [7, Lemma 3.3], $H^1_f(X_K, A) = 0$. Hence, the cohomology sequence arising from
\[ A[p^m] \overset{\cdot p^m}{\longrightarrow} A \overset{p^m}{\longrightarrow} A, \]
yields an isomorphism \( H^2_f(X_K, A[p^m]) \cong H^1_f(X_K, A)/p^m \). Consider the commutative diagram (with \( m_2 \geq m_1 \))

\[
\begin{array}{ccc}
H^1_f(X_K, A)/p^{m_1} & \longrightarrow & H^2_f(X_K, A[p^{m_1}]) \\
\downarrow & & \downarrow \\
H^1_f(X_K, A)/p^{m_2-m_1} & \longrightarrow & H^2_f(X_K, A[p^{m_2}])
\end{array}
\]

An element of \( H^1_f(X_K, A)/p^{m_1} \) of order \( p^r \) goes to zero via the vertical map on the left as soon as \( m_2 \geq m_1 + r \), hence the direct limit provides \( \lim_{m} H^1_f(X_K, A)/p^m = 0 \) and, eventually, \( \lim_{m} \ker^2_m = 0 \) as well. By (3.2)

\[
0 = \lim_{m} H^1(K_{\infty}/K, H^1_f(X_{K_{\infty}}, A[p^m])) = H^1(K_{\infty}/K, H^1_f(X_{K_{\infty}}, A[p^\infty])) ,
\]

which yields

\[
H^1_f(X_{K_{\infty}}, A[p^\infty]) = (\gamma p^{(K)} - 1)H^1_f(X_{K_{\infty}}, A[p^\infty]) = (\gamma - 1)H^1_f(X_{K_{\infty}}, A[p^\infty]) .
\]

We get the claim by taking the direct limit on the finite subextensions \( K \).

**Definition 3.4.** For any finite extension \( L/F \) we define the **Tate module of the Selmer group** of \( L \) to be

\[
T_p(Sel(L)) := \lim_n Sel_A(L)p^n = \lim_n \ker \left( H^1_f(X_L, A[p^n]) \to \prod_v H^1_f(X_{L_v}, A)[p^n] \right) .
\]

For any infinite extension \( \mathcal{L}/F \) the Tate module \( T_p(Sel(\mathcal{L})) \) is defined via inverse limit on the finite subextensions with respect to the corestriction maps.

**Lemma 3.5.** Let \( \mathcal{F}/F \) be a totally ramified extension, then \( T_p(Sel(\mathcal{F}_d)) \sim_{\Lambda(\mathcal{F}_d)} 0 \) for any \( d \geq 2 \), where \( \sim_{\Lambda(\mathcal{F}_d)} \) means pseudo-isomorphic \( \Lambda(\mathcal{F}_d) \)-modules.

**Proof.** We recall that \( F^{(p)} \) is the arithmetic \( \mathbb{Z}_p \)-extension of \( F \) and we denote by \( F_n^{(p)} \) its layers. Let \( F_n \) denote the layers of the \( \mathbb{Z}_p \)-extensions \( \mathcal{F}_d \): note that \( \text{Gal}(F_n/F) \simeq (\mathbb{Z}/p^n)^d \) and \( \mathcal{F} \) and \( F^{(p)} \) are disjoint. By [12, Theorem 1.7], \( Sel(F_n F^{(p)}) \) is a finitely generated torsion \( \Lambda(F^{(p)}) \)-module and this implies that the \( \mathbb{Z}_p \)-coranks of \( Sel(F_n F^{(p)}_t) \) are bounded (see, e.g., the proof of [1, Corollary 4.14]). Moreover for any \( s \geq t \), the restriction maps

\[
Sel_{\Lambda}(F_n F^{(p)}_t)_{p^m} \longrightarrow Sel_{\Lambda}(F_n F^{(p)}_s)_{p^m}
\]

have finite kernels (embedded in \( H^1(\text{Gal}(F_n^{(p)}_t/F), A[p^m](F_n F^{(p)})) \), by the analogue of diagram (2.1)) of order bounded by \( |H^1(\text{Gal}(F^{(p)}_t/F), A[p^\infty](F_n F^{(p)}))| \), which is finite by [2, Lemma 3.4]. Hence the inverse limit of those kernels (with respect to multiplication by powers of \( p \)) is 0 and the restriction map between Tate modules is injective.

Let \( t \) be such that the corank of \( Sel(F_n F^{(p)}_t) \) is maximal: then any \( \alpha \in T_p(Sel(F_n F^{(p)}_t)) \) \( (s \geq t) \) is represented by a torsion element modulo (the image of) \( T_p(Sel(F_n F^{(p)}_s)) \). The diagram

\[
\begin{array}{ccc}
T_p(Sel(F_n F^{(p)}_t)) \otimes_{\text{res}^{n,s}_t} & \longrightarrow & T_p(Sel(F_n F^{(p)}_s)) \\
\downarrow & \downarrow \text{cor}^{n,s}_{n,t} & \downarrow \text{res}^{n,s}_t \\
T_p(Sel(F_n F^{(p)}_t)) & \longrightarrow & T_p(Sel(F_n F^{(p)}_s))
\end{array}
\]
shows that
\[ \bigcap_{s>t} \operatorname{cor}_{n,t}^s \left( T_p(\text{Sel}(F_n F_s^{(p)})) \right) \subseteq \bigcap_{s>t} \operatorname{cor}_{n,t}^s \left( T_p(\text{Sel}(F_n F_s^{(p)}))_{\text{tor}} \right) + p^{s-t} T_p(\text{Sel}(F_n F_t^{(p)})) \]
\[ = \bigcap_{s>t} \operatorname{cor}_{n,t}^s \left( T_p(\text{Sel}(F_n F_s^{(p)}))_{\text{tor}} \right). \]

Via the Kummer sequence one has
\[ T_p(\text{Sel}(F_n F_s^{(p)})) = \lim_{s} T_p(\text{Sel}(F_n F_s^{(p)}))_{\text{tor}} \subseteq \lim_{s} T_p(\text{Sel}(F_n F_s^{(p)}))_{\text{tor}} \subseteq \lim_{s} A[p^\infty](F_n F_s^{(p)}). \]

Now using the layers \( F_n F_n^{(p)} \) for the \( \mathbb{Z}_p \)-extension \( \mathcal{F}_d F_n^{(p)}/F \), the formula
\[ \operatorname{cor}_{n,n}^m = \operatorname{cor}_{n,n}^m \circ \operatorname{cor}_{n,m}^m \]
and the previous computation, one has that
\[ T_p(\text{Sel}(\mathcal{F}_d)) = \lim_{n} T_p(\text{Sel}(F_n F_n^{(p)})) \subseteq \lim_{n} A[p^\infty](F_n F_n^{(p)}) \]
is a finitely generated \( \mathbb{Z}_p \)-module.

To conclude just note that the restriction maps
\[ \text{Sel}(F_n)^{p^n} \to \text{Sel}(F_n F_n^{(p)})^{p^n} \]
have kernels whose \( \mathbb{Z}_p \)-corank is bounded by the corank of \( H^1(\text{Gal}(F_n^{(p)}/F), A[p^\infty](\mathcal{F}_d F_n^{(p)})) \)
(note that, by [15, Proposition 2.11] this is often finite). Taking limits we have that
\[ T_p(\text{Sel}(\mathcal{F}_d)) = \lim_{n} \text{Sel}(F_n)^{p^n} \]
is a finitely generated \( \mathbb{Z}_p \)-module as well, hence \( A(\mathcal{F}_d) \)-pseudo-null for \( d \geq 2 \).

Now we are ready to deal with the module \( \mathcal{S}(\mathcal{F}_d)^{\Gamma_d^{-1}} \).

**Theorem 3.6.** Assume \( \mathcal{F}/F \) is a totally ramified extension. For any \( d \geq 3 \) we have
\[ \text{Ch}_{A(\mathcal{F}_{d-1})}(\mathcal{S}(\mathcal{F}_d)^{\Gamma_d^{-1}}) = (1). \]

**Proof.** Consider the following diagram
\[ \begin{array}{c}
\xymatrix{ \text{Sel}(\mathcal{F}_d) & H^1_j(\mathcal{X}_d, A[p^\infty]) \ar[d]_{\gamma^{-1}} & H^1(\mathcal{X}_d, A) \ar[d]_{\gamma^{-1}} & \text{Coker}(\phi_d) \ar[d]_{\gamma^{-1}} \\
\text{Sel}(\mathcal{F}_d) & H^1_j(\mathcal{X}_d, A[p^\infty]) \ar[d]_{\gamma^{-1}} & H^1(\mathcal{X}_d, A) \ar[d]_{\gamma^{-1}} & \text{Coker}(\phi_d) }
\end{array} \]
(where \( H^i(\mathcal{X}_d, A) := \prod_{w} H^i_f(\mathcal{X}_{d,w}, A[p^\infty]) \) and the surjectivity of the second vertical arrow comes from the previous lemma). Inserting \( \mathcal{M}(\mathcal{F}_d) := \text{Im}(\phi_d) \), we get two diagrams
\[ \begin{array}{c}
\xymatrix{ \text{Sel}(\mathcal{F}_d) & H^1_j(\mathcal{X}_d, A[p^\infty]) \ar[d]_{\gamma^{-1}} & \mathcal{M}(\mathcal{F}_d) & H^1(\mathcal{X}_d, A) \ar[d]_{\gamma^{-1}} & \text{Coker}(\phi_d) \\
\text{Sel}(\mathcal{F}_d) & H^1_j(\mathcal{X}_d, A[p^\infty]) \ar[d]_{\gamma^{-1}} & \mathcal{M}(\mathcal{F}_d) & H^1(\mathcal{X}_d, A) \ar[d]_{\gamma^{-1}} & \text{Coker}(\phi_d) }
\end{array} \]
(3.4)

From the snake lemma sequence of the first one, we obtain the isomorphism
\[ \mathcal{M}(\mathcal{F}_d)^{\Gamma_d^{-1}}/\text{Im}(\phi_d^{\Gamma_d^{-1}}) \simeq \text{Sel}(\mathcal{F}_d)/(\gamma - 1). \]

\[ \xymatrix{ \mathcal{M}(\mathcal{F}_d)^{\Gamma_d^{-1}}/\text{Im}(\phi_d^{\Gamma_d^{-1}}) \ar[r] & \text{Sel}(\mathcal{F}_d)/(\gamma - 1) }
\]
(where $\phi_d^{\Gamma_d-1}$ is the restriction of $\phi_d$ to $H^1_X(X_d, A[p^{\infty}])^{\Gamma_d-1}$. The snake lemma sequence of the second diagram (its “upper” row) yields an isomorphism

\begin{equation}
\mathcal{H}^1(X_d, A)^{\Gamma_d-1} / \mathcal{M}(\mathcal{F}_d)^{\Gamma_d-1} \simeq \text{Coker}(\phi_d)^{\Gamma_d-1}.
\end{equation}

The injection $\mathcal{M}(\mathcal{F}_d)^{\Gamma_d-1} \hookrightarrow \mathcal{H}^1(X_d, A)^{\Gamma_d-1}$ induces an exact sequence

$$
\mathcal{M}(\mathcal{F}_d)^{\Gamma_d-1} / \text{Im}(\phi_d^{\Gamma_d-1}) \hookrightarrow \mathcal{H}^1(X_d, A)^{\Gamma_d-1} / \text{Im}(\phi_d^{\Gamma_d-1}) \twoheadrightarrow \mathcal{H}^1(X_d, A)^{\Gamma_d-1} / \mathcal{M}(\mathcal{F}_d)^{\Gamma_d-1}
$$

(with a little abuse of notation we are considering $\text{Im}(\phi_d^{\Gamma_d-1})$ as a submodule of $\mathcal{H}^1(X_d, A)^{\Gamma_d-1}$ via the natural injection above) which, by (3.5) and (3.6), yields the sequence

\begin{equation}
\text{Sel}(\mathcal{F}_d)/ (\gamma - 1) \twoheadrightarrow \text{Coker}(\phi_d^{\Gamma_d-1}) \twoheadrightarrow \text{Coker}(\phi_d)^{\Gamma_d-1}.
\end{equation}

Now consider the following diagram

\[
\begin{array}{c}
\mathcal{H}^1(\Gamma_{\Gamma_d-1}, A[p^{\infty}]) \leftarrow H^1(\mathcal{F}_{\Gamma_d-1}, A[p^{\infty}]) \rightarrow H^1(\mathcal{F}_d, A)[p^{\infty}]^{\Gamma_d-1} \rightarrow 0 \\
\downarrow \phi_d^{\Gamma_d-1} \downarrow \phi_d \downarrow \phi_d^{\Gamma_d-1} \\
\mathcal{H}^1(\Gamma_{\Gamma_d-1}, A) \leftarrow \mathcal{H}^1(\mathcal{F}_{\Gamma_d-1}, A) \rightarrow \mathcal{H}^1(\mathcal{F}_d, A)[p^{\infty}]^{\Gamma_d-1} \rightarrow \mathcal{H}^2(\Gamma_{\Gamma_d-1}, A)
\end{array}
\]

where:

- the vertical maps are all induced by the product of restrictions;
- the horizontal lines are just the Hochschild-Serre sequences for global and local cohomology;
- the 0 in the upper right corner comes from $H^2(\Gamma_{\Gamma_d-1}, A[p^{\infty}]) = 0$;
- the surjectivity on the lower right corner comes from $\mathcal{H}^2(\mathcal{F}_{\Gamma_d-1}, A) = 0$, which is a direct consequence of $[10, \text{Theorem III.7.8}].$

This yields a sequence (from the snake lemma)

\begin{equation}
\text{Coker}(\phi_d-1) \rightarrow \text{Coker}(\phi_d^{\Gamma_d-1}) \rightarrow \mathcal{H}^2(\Gamma_{\Gamma_d-1}, A) = \prod_{w} H^2(\Gamma_{\Gamma_d-1, w}, A(\mathcal{F}_{\Gamma_d, w}))[p^{\infty}].
\end{equation}

**The module $\text{Coker}(\phi_d-1)$.** The Kummer map induces a surjection $H^1(\mathcal{F}_{\Gamma_d-1}, A[p^{\infty}]) \rightarrow H^1(\mathcal{F}_d, A)[p^{\infty}]$ which fits in the diagram

\[
\begin{array}{c}
H^1(\mathcal{F}_{\Gamma_d-1}, A[p^{\infty}]) \xrightarrow{\phi_d-1} H^1(\mathcal{F}_d, A) \\
\downarrow \lambda_{\Gamma_d-1} \\
H^1(\mathcal{F}_d, A)[p^{\infty}]
\end{array}
\]

($\lambda_{\Gamma_d-1}$ is again a product of restrictions). This yields surjective maps $\text{Im}(\phi_d-1) \rightarrow \text{Im}(\lambda_{\Gamma_d-1})$ and, eventually, $\text{Coker}(\lambda_{\Gamma_d-1}) \rightarrow \text{Coker}(\phi_d-1)$. For any finite extension $K/F$ we have a similar map

$$
\lambda_K : H^1(X_K, A)[p^{\infty}] \rightarrow H^1(X_K, A)
$$

whose cokernel verifies

$$
\text{Coker}(\lambda_K)^\vee \simeq T_p(\text{Sel}_{A'}(K)_p)
$$

(by [6, Main Theorem]), where $A'$ is the dual abelian variety of $A$ and $T_p$ denotes the $p$-adic Tate module.

Taking limits on all the finite subextensions of $\mathcal{F}_{\Gamma_d-1}$ (with respect to the corestriction maps) we find

$$
\text{Coker}(\lambda_{\Gamma_d-1})^\vee \simeq T_p(\text{Sel}_{A'}(\mathcal{F}_{\Gamma_d-1})_p) \sim_{\Lambda(\mathcal{F}_{\Gamma_d-1})} 0,
$$

by Lemma 3.5.
The modules $H^2(\Gamma_{d-1,w}^d, A(F_{d,w}))[[p^\infty]]$. If the prime splits completely in $F_d/F_{d-1}$, then obviously $H^2(\Gamma_{d-1,w}^d, A(F_{d,w}))[[p^\infty]] = 0$. If the place is ramified or inert, then $\Gamma_{d-1,w}^d \simeq \mathbb{Z}_p$. Consider the exact sequence

$$A(F_{d,w})[p] \xrightarrow{\pi_p} A(F_{d,w}) \xrightarrow{p} pA(F_{d,w})$$

which yields a surjection

$$H^2(\Gamma_{d-1,w}^d, A(F_{d,w})[p]) \xrightarrow{\pi_p} H^2(\Gamma_{d-1,w}^d, A(F_{d,w}))[p] .$$

The module on the left is trivial because $cd_p(\mathbb{Z}_p) = 1$, hence $H^2(\Gamma_{d-1,w}^d, A(F_{d,w}))[p] = 0$ and this yields $H^2(\Gamma_{d-1,w}^d, A(F_{d,w}))[p^\infty] = 0$.

The sequence (3.8) implies that $\text{Coker}(\phi_{\Gamma_{d-1}^d})$ is $\Lambda(F_{d-1})$ pseudo-null for $d \geq 3$ and, by (3.7), we get $\text{Sel}(F_d)/((\gamma - 1))$ is pseudo-null as well. Therefore

$$\text{Cl}_{\Lambda}(\text{Sel}(F_d)/((\gamma - 1))) = (1) .$$

\[\square\]

**Remark 3.7.** Assuming $d \geq \max\{e+1, 3\}$ (i.e., $\mathcal{S}(F_{d-1})$ is torsion) and using [15, Proposition 4.2] to deal with the Tate module, in place of the more general but weaker Lemma 3.5, one actually gets $\mathcal{S}(F_d)^{\Gamma_{d-1}^d} = 0$.

A direct consequence of equation (2.9) and Theorems 3.2 and 3.6 is

**Corollary 3.8.** Assume $F/F$ is a totally ramified extension, then, for any $d \gg 0$ and any $\mathbb{Z}_p$-subextension $F_d/F_{d-1}$, one has

$$\pi_{d-1}^{d}(\text{Cl}_{\Lambda}(\mathcal{S}(F))) = \text{Cl}_{\Lambda}(\mathcal{S}(F_{d-1})) .$$

The modules $\mathcal{S}(F_d)$ verify the hypotheses of Theorem 1.1 (because of Proposition 2.7 and Theorem 3.6), so we can define

**Definition 3.9.** For a totally ramified extension $F/F$, the pro-characteristic ideal of $\mathcal{S}(F)$ is

$$\tilde{\text{Cl}}_{\Lambda}(\mathcal{S}(F)) := \lim_{F_d} \text{Cl}_{\Lambda}(\mathcal{S}(F_d)) \subseteq \Lambda .$$

We remark that Definition 3.9 only depends on the extension $F/F$ and not on the filtration of $\mathbb{Z}_p^d$-extension we choose inside it. Indeed with two different filtrations $\{F_d\}$ and $\{F'_d\}$ we can define a third one by putting

$$\mathcal{F}_0' := F_d \text{ and } \mathcal{F}_n' = F_nF'_n \forall n \geq 1 .$$

By Corollary 3.8, the limits of the characteristic ideals of the filtrations we started with coincide with the limit on the filtration $\{\mathcal{F}_n\}$ (see [4, Remark 3.11] for an analogous statement for characteristic ideals of class groups).

This pro-characteristic ideal could play a role in the Iwasawa Main Conjecture (IMC) for a totally ramified extension of $F$ as the algebraic counterpart of a $p$-adic $L$-function associated to $A$ and $F$ (see [1, Section 5] or [3, Section 3] for similar statements but with Fitting ideals). Anyway, at present, the problem of formulating a (conjectural) description of this ideal in terms of a natural $p$-adic $L$-functions (i.e., a general non-noetherian Iwasawa Main Conjecture) is still wide open. However, we can say something if $A$ is already defined over the constant field of $F$.

**Theorem 3.10.** [Non-noetherian IMC for constant abelian varieties] Assume $A/F$ is a constant abelian variety and let $F/F$ be a totally ramified extension as above. Then there
exists an element $\theta_{A,F}$ interpolating the classical $L$-function $L(A, \chi, 1)$ (where $\chi$ varies among characters of $\text{Gal}(F/F)$) such that one has an equality of ideals in $\Lambda(F)$

$$\tilde{Ch}_{\Lambda(F)}(S(F)) = (\theta_{A,F}) \quad (3.10)$$

**Proof.** This is a simple consequence of [9, Theorem 1.3]. Namely, the element $\theta_{A,L}$ is defined in [9, Section 7.2.1] for any abelian extension $L/F$ unramified outside a finite set of places. It satisfies $\pi^{-1}_{d-1}(\theta_{A,F_d}) = \theta_{A,F_{d-1}}$ by construction and the interpolation formula (too complicated to report it here) is proved in [9, Theorem 7.3.1]. Since $A$ has good reduction everywhere, our results apply here and both sides of (3.10) are defined. Finally [9, Theorem 1.3] proves that $Ch_{\Lambda(F_d)}(S(F_d)) = (\theta_{A,F_d})$ for all $d$ and (3.10) follows by just taking a limit. \qed

**References**


