

SMOOTHNESS OF SUBRIEMANNIAN ISOMETRIES

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ABSTRACT. We show that the group of isometries (i.e., distance-preserving homeomorphisms) of an equiregular subRiemannian manifold is a finite-dimensional Lie group of smooth transformations. The proof is based on a new PDE argument, in the spirit of harmonic coordinates, establishing that in an arbitrary subRiemannian manifold there exists an open dense subset where all isometries are smooth.

1. INTRODUCTION

A classical result from 1939, due to Myers and Steenrod [MS39], states that any distance preserving homeomorphism (henceforth called isometry) between open subsets of smooth Riemannian manifolds is a smooth diffeomorphism. This regularity opens the way for the use of analytical tools in the study of metric properties of manifolds. Since 1939 several different proofs for this result have been proposed for Riemannian structures with varying degrees of smoothness. We refer the reader to the work of Palais [Pal57], Calabi and Hartman [CH70], and Taylor [Tay06]. The latter introduced a novel streamlined approach to the regularity issue, which (unlike many of the previous arguments) is not based on the study of the action of isometries on geodesics but rather relies on PDE. Namely, in [Tay06], Taylor notes that isometries preserve harmonic functions. Consequently, the composition of the isometry with harmonic coordinates must be harmonic and as smooth as the ambient geometry.

Since the distance preserving property transfers easily to Gromov-Hausdorff limits for sequences of pairs of Riemannian manifolds, in many applications it becomes relevant to study isometries between metric spaces obtained this way. In particular, in various settings including Mostow rigidity theory and the study of analysis and geometry on boundaries of strictly pseudo-convex domains in \mathbb{C}^n , the limit spaces fail to be Riemannian manifolds but have instead a natural subRiemannian structure. This gives rise to the problem of establishing regularity of isometries in this broader setting. Since subRiemannian distance functions are generally not smooth with respect to the underlying differential structure, this regularity problem is quite different from its Riemannian counterpart.

Throughout this paper we define a subRiemannian manifold as a triplet (M, Δ, g) where M is a connected, smooth manifold of dimension $n \in \mathbb{N}$, Δ denotes a subbundle of the tangent bundle TM that bracket generates TM , and g is a positive definite smooth, bilinear

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form defined on Δ , see [Mon02]. Analogously to the Riemannian setting, one can endow (M, Δ, g) with a metric space structure by defining the *Carnot-Caratheodory* (CC) control distance: For any pair $x, y \in M$ set

$$d(x, y) = \inf\{\delta > 0 \text{ such that there exists a curve } \gamma \in C^\infty([0, 1]; M) \text{ with endpoints } x, y \\ \text{such that } \dot{\gamma} \in \Delta(\gamma) \text{ and } |\dot{\gamma}|_g \leq \delta\}.$$

Curves whose velocity vector lies in Δ are called *horizontal*, their length is defined in an obvious way. A horizontal curve is a *geodesic* if it is locally distance minimizing. A geodesic is *normal* if it satisfies a subRiemannian analogue of the geodesic equation. One of the striking features of subRiemannian geometry is that not all geodesics are normal (see [Mon02] and references therein).

The first systematic study of distance preserving homeomorphisms in the subRiemannian setting goes back to the groundbreaking papers by Strichartz [Str86, Str89] and by Hamenstädt [Ham90]. Henceforth we will refer to distance preserving homeomorphisms as isometries and will specify if they are Riemannian or subRiemannian only when it is not completely clear from the context. It follows from Hamenstädt's results that, assuming that all geodesics are normal, isometries are smooth [Ham90, Theorem 6.2]. If the underlying structure is a Carnot group (see Example 2.4), then by virtue of the work of Hamenstädt and of Kishimoto [Ham90, Corollary 8.4], [Kis03, Theorem 4.2], one has a stronger result: Global isometries are compositions of group translations and group isomorphisms. More recently, Ottazzi and the second-named author [LDO14, Theorem 1.1] have established a local version of this result, that applies to isometries between open subsets of Carnot groups and that does not rely on the Hamenstädt-Kishimoto's arguments. Pansu's differentiability Theorem [Pan89] allows us to summarize the previous results as follows.

Theorem 1.1 ([LDO14]). *Let $\mathbb{G}_1, \mathbb{G}_2$ be Carnot groups and for $i = 1, 2$ consider $\Omega_i \subset \mathbb{G}_i$ open sets. If $f : \Omega_1 \rightarrow \Omega_2$ is an isometry then \mathbb{G}_1 is isomorphic to \mathbb{G}_2 and f is the restriction to Ω_1 of the composition of a group translation with a Lie group automorphism.*

Smoothness of isometries in this setting, where not all geodesics are normal, was settled for globally defined isometries by Kishimoto, using a theorem of Montgomery and Zippin [MZ74, page 208, Theorem 2], see also the detailed proof in [LDO14, Corollary 2.12], while in the local case (of mapping between open subsets of Carnot groups) it follows from the study of 1-quasiconformal mappings by Cowling and the first-named author [CC06].

The main results of the present paper represent an extension of the work mentioned above to the general subRiemannian setting, without the extra assumptions on the presence of homogeneous structures or on the absence of abnormal geodesics.

Theorem 1.2. *Isometries between subRiemannian manifolds are smooth in any open sets where the subRiemannian structure is equiregular. In particular, there exists an open dense subset in which every isometry is smooth.*

The proof of Theorem 1.2 proceeds along two steps of independent interest: Theorem 1.3 and Theorem 1.4.

Theorem 1.3. *Let $F : M \rightarrow N$ be an isometry between two subRiemannian manifolds. If there exist two C^∞ volume forms vol_M and vol_N such that $F_* \text{vol}_M = \text{vol}_N$, then F is a C^∞ diffeomorphism.*

A C^∞ volume form on an n -dimensional manifold is the measure associated to a C^∞ nonvanishing n -form on the manifold. When $F : M \rightarrow N$ is a continuous map, the measure $F_* \text{vol}_M$ is defined by $F_* \text{vol}_M(A) := \text{vol}_M(F^{-1}(A))$, for all measurable sets $A \subseteq N$.

The proof of Theorem 1.3 is similar in spirit to Taylor's approach to the regularity of Riemannian isometries via harmonic coordinates [Tay06]. However, it appears very difficult at present to establish existence of harmonic coordinates in the subRiemannian setting. To overcome this difficulty we observe that while isometries preserve harmonic functions (critical points of the Dirichlet energy (2.10)) they also preserve weak solutions of non-homogenous PDE in divergence form built by polarization from the Dirichlet energy, see Corollary 2.14. Using this observation we show that the pull-back of *any* system of coordinates in the target manifold satisfies a subelliptic linear PDE for which appropriate L^p estimates holds. Regularity then follows through a bootstrap argument.

Next we turn to the hypothesis of Theorem 1.3. The existence of smooth volume forms that are preserved by (metric) isometries is far from trivial. The natural choice for an isometric invariant measure on a manifold is the top-dimensional Hausdorff measure \mathcal{H} (or spherical Hausdorff measure \mathcal{S}). In the Riemannian setting \mathcal{H} and \mathcal{S} coincide with the Riemannian volume measure and hence they readily provide smooth measures that are isometric invariant. A similar argument carries out to the Carnot group setting, where the Hausdorff measures are Haar measures and hence equal up to a constant to any left-invariant Riemannian measure. However, for general *equiregular* subRiemannian manifolds (see Definition 2.2) the counterpart of this statement is false. In [Mit85], Mitchell shows that if Q denotes the local homogeneous dimension (see (2.3)), then the density of the Lebesgue measure with respect to the Q -dimensional Hausdorff volume is bounded from above and below by positive constants. In their recent paper [ABB12], Agrachev, Barillari, and Boscain have studied the density function of any smooth volume form with respect to the spherical Hausdorff volume \mathcal{S} in *equiregular* subRiemannian manifolds and have proved that it is always continuous. Moreover, if the topological dimension of M is less or equal to 4 this function is smooth. They also showed that for dimensions 5 and higher this is no longer true and exhibit examples where the density is C^3 but not C^5 .

In view of this obstacle one has to find a different approach. Namely, we consider a canonical smooth volume form that is defined only in terms of the subRiemannian structure, the so-called *Popp measure* (see [Mon02]) and show the following result.

Theorem 1.4. *Let $F : M \rightarrow N$ be an isometry between equiregular subRiemannian manifolds. If vol_M and vol_N are the Popp measures on M and N , respectively, then $F_* \text{vol}_M = \text{vol}_N$.*

For smooth isometries a proof of the above result can be find in [BR13]. The novelty in Theorem 1.4 consists in the fact that we are considering isometries with no smoothness assumptions.

The proof of Theorem 1.4 is based on the nilpotent tangent space approximation of the subRiemannian structure (see Section 2) and on the detailed analysis of the induced

map among such tangent spaces. Ultimately, the argument rests on Theorem 1.1 and on Theorem 2.6, which is a representation formula for Hausdorff measures recently established in [ABB12] and [GJ14, Section 3.2, Remark 2].

Corollary 1.5. *Every $F : M \rightarrow N$ isometry between equiregular subRiemannian manifolds is a C^∞ diffeomorphism.*

Theorem 1.2 follows immediately from Corollary 1.5 once we notice that, if M is a subRiemannian manifold, then there exists an open dense subset $U \subset M$ such that the subRiemannian structure restricted to any connected component of U is equiregular.

We conclude this introduction with some consequences of our regularity result, which in fact were our initial motivations. Note that one can always extend an equiregular subRiemannian structure to a Riemannian one, i.e., there exists a Riemannian metric tensor on TM that coincides with g when restricted to Δ . Riemannian extensions are also called Riemannian metrics that *tame* the subRiemannian metric, [Mon02, Definition 1.9.1]. Hence, suppose that $F : M \rightarrow N$ is an isometry between two equiregular subRiemannian manifolds. For any Riemannian extension g_M , the push-forward $g_N := F_*g_M$ is well-defined because of Corollary 1.5 and tames the subRiemannian structure on N . By construction, the map $F : (M, g_M) \rightarrow (N, g_N)$ is also a Riemannian isometry.

It is less clear that one can find a single Riemannian extension for which a subRiemannian self-isometry is also a Riemannian isometry. This is the content of our main application: a structure theorem on the group of subRiemannian self-isometries $\text{Isom}(M)$ and its correspondence to the group of Riemannian isometries associated to a Riemannian metric that tames the subRiemannian one.

Theorem 1.6. *If M is an equiregular subRiemannian manifold, then*

- (i) $\text{Isom}(M)$ admits a structure of finite-dimensional Lie group;
- (ii) for all compact subgroup $K < \text{Isom}(M)$, there exists a Riemannian extension g_K on M such that $K < \text{Isom}(M, g_K)$.

In particular, since the group of isometries that fixes a point $p \in M$, denoted by $\text{Isom}_p(M)$, is compact (in view of Ascoli-Arzelà's theorem) then we have the following immediate consequence.

Corollary 1.7. *If M is an equiregular subRiemannian manifold, then for all $p \in M$, there exists a Riemannian extension \hat{g} on M such that $\text{Isom}_p(M) < \text{Isom}(M, \hat{g})$.*

A further consequence is based on an argument in [LDO14]. Just like Riemannian isometries, subRiemannian isometries are completely determined by the value and the horizontal differential at a point.

Corollary 1.8. *Let M and N be two connected equiregular subRiemannian manifolds. Let $p \in M$ and let Δ be the horizontal bundle of M . Let $f, g : M \rightarrow N$ be two isometries. If $f(p) = g(p)$ and $df|_{\Delta_p} = dg|_{\Delta_p}$, then $f = g$.*

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2. BASIC DEFINITIONS AND PRELIMINARY RESULTS

Consider a subRiemannian manifold (M, Δ, g) and iteratively set $\Delta^1 := \Delta$, and $\Delta^{i+1} := \Delta^i + [\Delta^i, \Delta]$ for $i \in \mathbb{N}$. The bracket generating condition (also called *Hörmander's finite rank hypothesis*) is expressed by the existence of $m \in \mathbb{N}$ such that, for all $p \in M$, one has

$$(2.1) \quad \Delta_p^m = T_p M.$$

Definition 2.2. A subRiemannian manifold (M, Δ, g) is *equiregular* if, for all $i \in \mathbb{N}$, the dimension of Δ_p^i is constant in $p \in M$. In this case, the *homogenous dimension* is

$$(2.3) \quad Q := \sum_{i=1}^{m-1} i [\dim(\Delta_p^i) - \dim(\Delta_p^{i-1})].$$

Consider the metric space (M, d) where (M, Δ, g) is an equiregular subRiemannian manifold and d is the corresponding control metric. As a consequence of Chow-Rashevsky Theorem such a distance is always finite and induces on M the original topology. As a result of Mitchell [Mit85], the Hausdorff dimension of (M, d) coincide with (2.3).

Example 2.4. *Carnot groups* are important examples of equiregular subRiemannian manifolds. Classical references are [Fol75, RS76, FS82]. These are simply connected analytic Lie groups \mathbb{G} whose Lie algebras \mathfrak{g} has a stratification $\mathfrak{g} = V^1 \oplus \dots \oplus V^m$ with $[V^1, V^j] = V^{1+j}$ and $[V^j, V^m] = 0$ for all $j = 1, \dots, m$. The corresponding subRiemannian structures are given by setting Δ to be the left-invariant subbundle associated to V^1 and by any choice of g left-invariant metric on Δ .

2.1. Tangent cones and measures. The *tangent cone* of the metric space (M, d) at a point $p \in M$ is the Gromov-Hausdorff limit $\mathcal{N}_p(M) := \lim_{t \rightarrow 0} (M, d/t, p)$. In view of Mitchell's work [Mit85] the metric space $\mathcal{N}_p(M)$ is described by the nilpotent approximation associated to the spaces Δ_p^i . In particular, $\mathcal{N}_p(M)$ is a Carnot group.

For the definition of Popp measure on a subRiemannian manifold (M, Δ, g) we refer the reader to [Mon02, Section 10.6]. Here we only recall that if $U \subset M$ is an open set upon which the subRiemannian structure is equiregular then the Popp measure is a smooth volume form on U .

Following [ABB12], we recall that, given any C^∞ volume form vol_M on M , e.g., the Popp measure, we have a C^∞ volume form induced by vol_M on the nilpotent approximation $\mathcal{N}_p(M)$, which we denote by $\mathcal{N}_p(\text{vol}_M)$. Moreover, if vol_M is the Popp measure on M and $\text{vol}_{\mathcal{N}_p(M)}$ is the Popp measure on $\mathcal{N}_p(M)$, then

$$(2.5) \quad \mathcal{N}_p(\text{vol}_M) = \text{vol}_{\mathcal{N}_p(M)}.$$

We shall use in a very crucial way the following representation formula for smooth volume measures.

Theorem 2.6 ([ABB12, pages 358–359], [GJ14, Section 3.2]). *Denote by Q the Hausdorff dimension of an equiregular subRiemannian manifold M and by \mathcal{S}_M^Q the spherical Hausdorff measure on M . Any C^∞ volume form is related to \mathcal{S}^Q by*

$$(2.7) \quad d\text{vol}_M = 2^{-Q} \mathcal{N}_p(\text{vol}_M)(B_{\mathcal{N}_p(M)}(e, 1)) d\mathcal{S}_M^Q.$$

Here we denote by $B_{\mathcal{N}_p(M)}(e, 1)$ the unit ball in the metric space $\mathcal{N}_p(M)$ with center the identity element.

Remark 2.8. When read in charts, smooth volume forms are absolutely continuous with respect to the Lebesgue measure. The volume density is smooth and bounded away from zero and infinity on compact sets. Namely, for any smooth volume form vol , one has $d\text{vol} = \eta d\mathcal{L}^n$, with η a smooth, strictly positive density function. Here and hereafter, by \mathcal{L}^n we denote the Lebesgue measure. In particular, there is no ambiguity to say that some property holds almost everywhere without specifying with respect to which of the two possible measures: a volume form vol or the Lebesgue measure.

2.2. Dirichlet energy. Let $X = \{X_1, \dots, X_r\}$ be an orthonormal frame of Δ in a subRiemannian manifold (M, Δ, g) , and vol a smooth volume measure on M . Denote by $\text{Lip}(M)$ the space of all Lipschitz functions with respect to the subRiemannian distance and recall the definition of the *horizontal gradient*

$$(2.9) \quad \nabla_H u := (X_1 u) X_1 + \dots + (X_r u) X_r,$$

defined a.e., as an $L_{\text{loc}}^\infty(M)$ function, for $u \in \text{Lip}(M)$ (see [GN98, Theorem 1.3]). Given a compact set $K \subseteq M$, we consider the (*horizontal Dirichlet*) energy (on K and with respect to vol) as

$$(2.10) \quad E(u) := E_{H,K}^{\text{vol}}(u) := \int_K \|\nabla_H u\|^2 d\text{vol},$$

and the associated quadratic form defined as, for all $u, v \in \text{Lip}(M)$,

$$(2.11) \quad E(u, v) := E_{H,K}^{\text{vol}}(u, v) := \int_K \langle \nabla_H u, \nabla_H v \rangle d\text{vol}.$$

Note that $2E(u, v) = E(u + v) - E(u) - E(v)$. The definitions above extend immediately to the larger class $W_{H,\text{loc}}^{1,2}$, as defined in Definition 2.15.

Regarding the next result and an introduction to the notion of (minimal) upper gradients, we refer the reader to Hajlasz and Koskela's monograph.

Lemma 2.12 ([HK00, page 51 and Section 11.2]). *If $u \in \text{Lip}(M)$, then $\|\nabla_H u\|$ coincides almost everywhere with the minimal upper gradient of u .*

Proposition 2.13. *Let $F : M \rightarrow N$ be an isometry between two subRiemannian manifolds. If vol_M and vol_N are two C^∞ volume forms such that $F_* \text{vol}_M = \text{vol}_N$ then, for all compact sets $K \subseteq M$,*

- (i) $E_{H,K}^{\text{vol}_M}(u \circ F) = E_{H,F(K)}^{\text{vol}_N}(u), \quad \forall u \in \text{Lip}(N);$
- (ii) $E_{H,K}^{\text{vol}_M}(u \circ F, v \circ F) = E_{H,F(K)}^{\text{vol}_N}(u, v), \quad \forall u, v \in \text{Lip}(N).$

Proof. Since F is distance and volume preserving then it is an isomorphism between the metric measure spaces (M, d_M, vol_M) and (N, d_N, vol_N) . In particular, since minimal upper gradients are a purely metric measure notion, the map F induces a one-to-one correspondence between minimal upper gradients between the two spaces. Hence, by Lemma 2.12, being $\|\nabla_H u\| \circ F$ the pull back to M of the minimal upper gradient of u in (N, d_N, vol_N) , it coincides a.e. with $\|\nabla_H(u \circ F)\|$, i.e., the minimal upper gradient of $u \circ F$ in (M, d_M, vol_M) . Consequently,

$$\int_K \|\nabla_H(u \circ F)\|^2 \, d\text{vol}_M = \int_K \|\nabla_H u\|^2 \circ F \, d\text{vol}_M = \int_{F(K)} \|\nabla_H u\|^2 \, d\text{vol}_N.$$

Since $E(\cdot)$ determines $E(\cdot, \cdot)$, the proof is concluded. \square

We denote by $\text{Lip}_{\text{loc}}(M)$ the functions on M that are Lipschitz on bounded sets. We denote by $\text{Lip}_c(M)$ the Lipschitz functions on M that have compact support.

Corollary 2.14. *If $u \in \text{Lip}_{\text{loc}}(N)$ and $g \in L^2(N)$ solve*

$$\int_N \langle \nabla_H u, \nabla_H v \rangle \, d\text{vol}_N = \int_N g v \, d\text{vol}_N, \quad \forall v \in \text{Lip}_c(N),$$

then the functions $\tilde{u} := u \circ F$ and $\tilde{g} := g \circ F$ solve

$$\int_M \langle \nabla_H \tilde{u}, \nabla_H v \rangle \, d\text{vol}_M = \int_M \tilde{g} v \, d\text{vol}_M, \quad \forall v \in \text{Lip}_c(M).$$

Proof. Note that, since F preserves compact sets, it gives a one-to-one correspondence $v \mapsto v \circ F$ between $\text{Lip}_c(N)$ and $\text{Lip}_c(M)$. Hence, any map in $\text{Lip}_c(M)$ is of the form $v \circ F$. Let $F(K)$ be a compact set containing the support of v . Then

$$\begin{aligned} \int_M \langle \nabla_H \tilde{u}, \nabla_H(v \circ F) \rangle \, d\text{vol}_M &= E_{H,K}^{\text{vol}_M}(u \circ F, v \circ F) \\ &= E_{H,F(K)}^{\text{vol}_N}(u, v) \\ &= \int_N \langle \nabla_H u, \nabla_H v \rangle \, d\text{vol}_N \\ &= \int_N g v \, d\text{vol}_N \\ &= \int_M (g \circ F)(v \circ F) \, d\text{vol}_M. \end{aligned}$$

\square

2.3. L^p estimates for subLaplacians. The regularity of isometries in Theorem 1.3 is based on a bootstrap argument, which eventually rests on certain L^p regularity estimates established by Rothschild and Stein, see [RS76, Theorem 18]. In this section we recall these estimates in Theorem 2.19, along with the basic definitions of horizontal Sobolev spaces.

Definition 2.15. Let $X = \{X_1, \dots, X_r\}$ be a system of smooth vector fields in \mathbb{R}^n satisfying Hörmander's finite rank hypothesis (2.1) and denote by \mathcal{L}^n the Lebesgue measure. For $k \in \mathbb{N}$ and for any multi-index $I = (i_1, \dots, i_k) \in \{1, \dots, r\}^k$ we define $|I| = k$ and $X^I u = X_{i_1} \dots X_{i_k} u$. For $p \in [1, \infty)$ we define the *horizontal Sobolev space* $W_H^{k,p}(\mathbb{R}^n, \mathcal{L}^n)$ associated to X to be the space of all $u \in L^p(\mathbb{R}^n, \mathcal{L}^n)$ whose distributional derivatives $X^I u$ are also in $L^p(\mathbb{R}^n, \mathcal{L}^n)$

for all multi-indexes $|I| \leq k$. This space can also be defined as the closure of the space of $C_c^\infty(\mathbb{R}^n)$ functions with compact support with respect to the norm

$$(2.16) \quad \|u\|_{W_H^{k,p}}^p := \|u\|_{L^p(\mathbb{R}^n, \mathcal{L}^n)}^p + \int_{\mathbb{R}^n} [\sum_{|I|=1}^k (X^I u)^2]^{p/2} d\mathcal{L}^n,$$

see [GN98], [FSSC96] and references therein. A function $u \in L^p(\mathbb{R}^n, \mathcal{L}^n)$ is in the local Sobolev space $W_{H,\text{loc}}^{k,p}(\mathbb{R}^n, \mathcal{L}^n)$ if, for any $\phi \in C_c^\infty(\mathbb{R}^n)$, one has $u\phi \in W_H^{k,p}(\mathbb{R}^n, \mathcal{L}^n)$.

The definition above extends *verbatim* to a subRiemannian manifold (M, Δ, g) endowed with a smooth volume measure vol_M . The $W_H^{k,p}$ Sobolev norm in this setting is defined through a frame $X = \{X_1, \dots, X_r\}$ of Δ by

$$(2.17) \quad \|u\|_{W_H^{k,p}}^p := \|u\|_{L^p(M, \text{vol}_M)}^p + \int_M [\sum_{|I|=1}^k (X^I u)^2]^{p/2} d\text{vol}_M.$$

Remark 2.18. Although the Sobolev norm (2.17) depends on the specific frame and smooth volume form chosen, the class $W_{H,\text{loc}}^{k,p}(M, \text{vol})$ does not. This is easily seen noticing that different choices of frames and smooth volume forms give rise to equivalent norms (on compact sets). In particular, one can check whether a function is in $W_{H,\text{loc}}^{k,p}$ through a smooth coordinate chart and using the Lebesgue measure, as in (2.16).

Theorem 2.19 ([RS76, Theorem 18]). *Let X_0, X_1, \dots, X_r be a system of smooth vector fields in \mathbb{R}^n satisfying Hörmander's finite rank hypothesis (2.1). Let*

$$(2.20) \quad \mathcal{L} := \sum_{i=1}^r X_i^2 + X_0$$

and consider a distributional solution to the equation $\mathcal{L}u = f$ in \mathbb{R}^n . For every $k \in \mathbb{N} \cup \{0\}$ and $1 < p < \infty$, if $f \in W_H^{k,p}(\mathbb{R}^n, \mathcal{L}^n)$ then $u \in W_{H,\text{loc}}^{k+2,p}(\mathbb{R}^n, \mathcal{L}^n)$.

Remark 2.21. If $u \in \text{Lip}_{\text{loc}}(\mathbb{R}^n)$ then $u \in W_{H,\text{loc}}^{1,p}(\mathbb{R}^n, \mathcal{L}^n)$ for all $p \in [1, \infty)$. In fact, by virtue of [GN98, Theorem 1.5] one has that the distributional derivatives $X_i u$ are in $L_{\text{loc}}^\infty(\mathbb{R}^n)$.

A standard argument, involving coordinate charts and cut-off functions, leads to a local version of the Rothschild-Stein estimates in any subRiemannian manifold. For the reader's convenience we include the proof.

Corollary 2.22. *Let M be a subRiemannian manifold equipped with a C^∞ volume form vol_M and let $k \in \mathbb{N}$ and $1 \leq p < \infty$. If $u \in W_{H,\text{loc}}^{k,p}(M, \text{vol}_M) \cap \text{Lip}_{\text{loc}}(M)$ is a solution of*

$$(2.23) \quad \int_M \langle \nabla_H u, \nabla_H v \rangle d\text{vol}_M = \int_M g v d\text{vol}_M, \quad \forall v \in \text{Lip}_c(M).$$

for some $g \in W_{H,\text{loc}}^{k,p}(M, \text{vol}_M)$, then $u \in W_{H,\text{loc}}^{k+1,p}(M, \text{vol}_M)$.

Proof. Consider an orthonormal frame $X = \{X_1, \dots, X_r\}$ of Δ and a smooth coordinate chart $\phi : U \rightarrow V$, with $U \subset M$ and $V \subset \mathbb{R}^n$. Let $v \in C_c^\infty(M)$ with support in U . Without loss of generality we can assume that both U and V are relatively compact. Setting

$$\tilde{\Delta} := \phi_* \Delta, \quad \tilde{X} := \phi_* X, \quad \tilde{u} := \phi_* u, \quad \tilde{v} := \phi_* v, \quad \tilde{g} := \phi_* g, \quad \tilde{\text{vol}}_M := \phi_* \text{vol}_M,$$

then through a change of variable, (2.23) reads as

$$\int_V \sum_{i=1}^r \tilde{X}_i \tilde{u} \tilde{X}_i \tilde{v} d\tilde{\text{vol}}_M = \int_V \tilde{g} \tilde{v} d\tilde{\text{vol}}_M.$$

Since vol_M is a smooth volume measure, we can write $d\text{vol}_M = \eta d\mathcal{L}^n$, for some C^∞ function $\eta : V \rightarrow \mathbb{R}$ bounded away from zero. Set

$$Y_1 := \sqrt{\eta} \tilde{X}_1, \dots, Y_r := \sqrt{\eta} \tilde{X}_r,$$

and note that the vector fields Y_1, \dots, Y_r are a (orthogonal) frame for $\tilde{\Delta}$, and in particular they satisfy the Hörmander finite rank condition (2.1). Since compactly supported smooth functions in U are in a one to one correspondence with compactly supported smooth functions on V under the action of ϕ , it is immediate that

$$\int_V \sum_{i=1}^r Y_i \tilde{u} Y_i v d\mathcal{L}^n = \int_V \eta \tilde{g} v d\mathcal{L}^n, \quad \forall v \in C_c^\infty(V),$$

that is, \tilde{u} is a distributional solution of the PDE

$$(2.24) \quad \mathcal{L}\tilde{u} = \sum_{j=1}^r Y_j^*(Y_j(\tilde{u})) = \eta \tilde{g}.$$

Here, if $Y_i = \sum_{j=1}^n a_{ij} \partial_j$, then Y_i^* denotes the *dual* vector (with respect to the Lebesgue measure) $Y_i^* = -\sum_{j=1}^n a_{ij} \partial_j - \sum_{j=1}^n \partial_j a_{ij}$. Consequently the left hand side of (2.24) is an operator \mathcal{L} of the form (2.20) with $X_0 = \sum_{i=1}^r \sum_{j=1}^n \partial_j a_{ij} Y_i \in \tilde{\Delta}$. By virtue of Remark 2.18 the functions \tilde{u} , and $\tilde{g}\eta$ are in the Sobolev space $W_{H,\text{loc}}^{k,p}(U, \mathcal{L}^n)$ corresponding to the Hörmander vector fields Y_1, \dots, Y_r . Next, in order to apply Theorem 2.19 we actually need to have the right hand side in (2.24) to belong to the smaller Sobolev space $W_H^{k,p}(\mathbb{R}^n, \mathcal{L}^n)$. To achieve this we consider for every $p \in U$ a cut-off function $\varphi \in C_c^\infty(U)$ which is identically one near p . It will be then enough to prove that $u\varphi \in W_H^{k+1,p}(\mathbb{R}^n, \mathcal{L}^n)$. Evaluating $\mathcal{L}(u\varphi)$ yields

$$\mathcal{L}(u\varphi) = \varphi \mathcal{L}u + \sum_{i=1}^r b_i Y_i u + bu,$$

where $b_i, b \in C_c^\infty(U)$. Since $b\tilde{u}, \varphi \mathcal{L}\tilde{u} \in W_H^{k,p}(\mathbb{R}^n, \mathcal{L}^n) \subset W_H^{k-1,p}(\mathbb{R}^n, \mathcal{L}^n)$ and $b_i Y_i u \in W_H^{k-1,p}(\mathbb{R}^n, \mathcal{L}^n)$ then the right hand side of the latter equation is in $W_H^{k-1,p}(\mathbb{R}^n, \mathcal{L}^n)$ for all $p \in [1, \infty)$. The conclusion now follows from Theorem 2.19 and from applying again Remark 2.18. \square

Smoothness of Sobolev functions is guaranteed by a Morrey-Campanato type embedding in an appropriate Hölder class.

Definition 2.25. Let $X = \{X_1, \dots, X_r\}$ be a system of smooth vector fields in \mathbb{R}^n satisfying Hörmander's finite rank hypothesis (2.1) inducing a control distance d . For $\alpha \in (0, 1)$ define the (Folland-Stein) Hölder class

$$C_{\text{loc}}^\alpha(\mathbb{R}^n) := \{u : \mathbb{R}^n \rightarrow \mathbb{R} \mid \forall \text{ compact } K, \exists C > 0 : \forall x, y \in K, |u(x) - u(y)| \leq Cd^\alpha(x, y)\}.$$

For any system of smooth vector fields satisfying Hörmander's finite rank hypothesis and for any bounded open set $\Omega \subset \mathbb{R}^n$, one can find constants $Q = Q(\Omega, X) > 0$, the so-called *local homogenous dimension* relative to X and Ω , $C = C(\Omega, X) > 0$ and $R = R(\Omega, X) > 0$, such that, for all metric balls centered in Ω with radius less than R , one has a lower bound on the Lebesgue volume of these balls in terms of a power of their radius, i.e., $\mathcal{L}^n(B) \geq C \text{diam}(B)^Q$. The quantity Q arises out of the celebrated doubling property established by Nagel, Stein, and Wainger in [NSW85] and plays an important role in the theory of Sobolev spaces in the subRiemannian setting. In case X is an equiregular distribution then Q does not depend on Ω and coincides with the homogenous dimension defined earlier. By virtue of Jerison's Poicaré inequality [Jer86], if $u \in W_{\text{H},\text{loc}}^{1,p}(\mathbb{R}^n)$ and $p > Q > 1$, then, for any d -metric ball $B \subset \mathbb{R}^n$, one has

$$(2.26) \quad \frac{1}{\mathcal{L}^n(B)} \int_B |u - u_B|^Q d\mathcal{L}^n \leq C \text{diam}(B)^{Q(\frac{p-Q}{p})},$$

where u_B denotes the average of u over B and $C > 0$ is a constant depending only on p, Q, X , and $\|u\|_{L^p(B)}$. A simple argument, see for instance [Lu98] and [Cap99, Theorem 6.9], starting from (2.26) leads to the following Morrey-Campanato type embedding.

Proposition 2.27. *Let $X = \{X_1, \dots, X_r\}$ be a system of smooth vector fields in \mathbb{R}^n satisfying Hörmander's finite rank hypothesis (2.1). For every bounded open set $\Omega \subset \mathbb{R}^n$ there exists a positive number Q depending only on n, X and Ω such that, if $p > Q$, then*

$$W_{\text{H},\text{loc}}^{1,p}(\Omega, \mathcal{L}^n) \subset C_{\text{loc}}^\alpha(\Omega)$$

with $\alpha = (p - Q)/p$.

3. PROOFS

3.1. Proof of Theorem 1.4. We begin by establishing the result in the special case of Carnot groups. Assume that M, N are two Carnot groups and let $G : M \rightarrow N$ be an isometry fixing the identity. In view of Theorem 1.1 the map G is a group isomorphism. Since on Carnot groups the Popp measures are left-invariant, $G_*(\text{vol}_M)$ is a Haar measure, hence it is a constant multiple of vol_N . Since G is an isometry, the constant is one.

Let us go back to the general case of equiregular subRiemannian manifolds. Recall the definition of nilpotent approximations $\mathcal{N}_p(M)$ from Section 2.1. We observe that, for every $p \in M$, since (M, p) and $(N, F(p))$ are isometric as pointed metric spaces, then, by uniqueness of Gromov-Hausdorff limits, there exists a (not necessarily unique) isometry $G_p : \mathcal{N}_p(M) \rightarrow \mathcal{N}_{F(p)}(N)$, fixing the identity. In particular, the unit balls of the tangents are the same, i.e.,

$$(3.1) \quad G_p(B_{\mathcal{N}_p(M)}(e, 1)) = B_{\mathcal{N}_{F(p)}(N)}(e, 1).$$

In view of the argument at the beginning of the proof, we have that the Popp measures are the same, i.e.,

$$(3.2) \quad (G_p)_*(\text{vol}_{\mathcal{N}_p(M)}) = \text{vol}_{\mathcal{N}_{F(p)}(N)}.$$

In addition, the spherical Hausdorff measures are preserved by isometries. Namely, we have

$$(3.3) \quad F_* \mathcal{S}_M^Q = \mathcal{S}_N^Q.$$

Therefore, for all measurable sets $A \subseteq N$, we have

$$\begin{aligned} F_* \text{vol}_M(A) &= \text{vol}_M(F^{-1}(A)) \\ (\text{from (2.7)}) \quad &= \frac{1}{2^Q} \int_{F^{-1}(A)} \mathcal{N}_p(\text{vol}_M)(B_{\mathcal{N}_p(M)}(e, 1)) d\mathcal{S}_M^Q(p) \\ (\text{from (3.1)}) \quad &= \frac{1}{2^Q} \int_{F^{-1}(A)} \mathcal{N}_p(\text{vol}_M)(G_p^{-1}(B_{\mathcal{N}_{F(p)}(N)}(e, 1))) d\mathcal{S}_M^Q(p) \\ &= \frac{1}{2^Q} \int_{F^{-1}(A)} (G_p)_* \mathcal{N}_p(\text{vol}_M)(B_{\mathcal{N}_{F(p)}(N)}(e, 1)) d\mathcal{S}_M^Q(p) \\ (\text{from (2.5)}) \quad &= \frac{1}{2^Q} \int_{F^{-1}(A)} (G_p)_* \text{vol}_{\mathcal{N}_p(M)}(B_{\mathcal{N}_{F(p)}(N)}(e, 1)) d\mathcal{S}_M^Q(p) \\ (\text{from (3.2)}) \quad &= \frac{1}{2^Q} \int_{F^{-1}(A)} \text{vol}_{\mathcal{N}_{F(p)}(N)}(B_{\mathcal{N}_{F(p)}(N)}(e, 1)) d\mathcal{S}_M^Q(p) \\ (q = F(p) \text{ and from (3.3)}) \quad &= \frac{1}{2^Q} \int_A \text{vol}_{\mathcal{N}_q(N)}(B_{\mathcal{N}_q(N)}(e, 1)) d\mathcal{S}_N^Q(q) \\ (\text{from (2.5)}) \quad &= \frac{1}{2^Q} \int_A \mathcal{N}_q(\text{vol}_N)(B_{\mathcal{N}_q(N)}(e, 1)) d\mathcal{S}_N^Q(q) \\ (\text{from (2.7)}) \quad &= \text{vol}_N(A). \end{aligned}$$

The proof is concluded. \square

Remark 3.4. We observe that, in view of Margulis-Mostow's version of Rademacher Theorem [MM95], in the proof above there is a natural choice for the maps G_p for almost every p .

3.2. The iteration step. Theorem 1.3 will be proved bootstrapping the following two results.

Proposition 3.5. *Let M and N be two subRiemannian manifolds equipped with C^∞ volume measures vol_M and vol_N . Let $F : M \rightarrow N$ be an isometry such that $F_* \text{vol}_M = \text{vol}_N$. Let $g \in C^\infty(N)$. Assume that*

- (i) $g \circ F \in W_{H,\text{loc}}^{k,p}(M, \text{vol}_M)$, for some $k \in \mathbb{N}$ and some $1 \leq p < \infty$.
- (ii) *The function $u \in W_{H,\text{loc}}^{k,p}(M, \text{vol}_M) \cap \text{Lip}(N, \mathbb{R})$ is a solution of*

$$\int_N \langle \nabla_H u, \nabla_H v \rangle d\text{vol}_N = \int_N g v d\text{vol}_N, \quad \forall v \in \text{Lip}_c(N).$$

Then $u \circ F \in W_{H,\text{loc}}^{k+1,p}(M, \text{vol}_M)$ and solves

$$\int_M \langle \nabla_H(u \circ F), \nabla_H v \rangle \, d\text{vol}_M = \int_M (g \circ F) v \, d\text{vol}_M, \quad \forall v \in \text{Lip}_c(M).$$

Proof. The last statement is an immediate consequence of Corollary 2.14. Since $g \circ F \in W_{H,\text{loc}}^{k,p}(M, \text{vol}_M)$ then the conclusion follows from Corollary 2.22. \square

Proposition 3.6. *Let U and V be two open sets of \mathbb{R}^n and $X = \{X_1, \dots, X_r\}$ a frame of smooth Hörmander vector fields in U . For $k \in \mathbb{N} \cup \{0\}$ and $1 \leq p < \infty$, denote by $W_{H,\text{loc}}^{k,p}(U, \mathcal{L}^n)$ the corresponding Sobolev space. Let $F = (F_1, \dots, F_n) : U \rightarrow V$ be a map such that, for all $i = 1, \dots, n$, one has $F_i \in W_{H,\text{loc}}^{k,p}(U, \mathcal{L}^n)$. If $g \in C^\infty(V)$, then $g \circ F \in W_{H,\text{loc}}^{k,p}(U, \mathcal{L}^n)$.*

Proof. For all $i = 1, \dots, n$ and $j = 1, \dots, r$, we have $X_j F_i \in W_{H,\text{loc}}^{k-1,p}(U, \mathcal{L}^n)$. Using the scalar chain rule for Sobolev functions one obtains

$$(3.7) \quad X_j(g \circ F) = X_j(g(F_1, \dots, F_n)) = \partial_1 g X_j F_1 + \dots + \partial_n g X_j F_n,$$

a.e. in U . Since $\partial_i g \in C^\infty$, then the right hand side of (3.7) is also in $W_{H,\text{loc}}^{k-1,p}(U, \mathcal{L}^n)$, concluding the proof. \square

3.3. Proof of Theorem 1.3. We work in (smooth) coordinates, so that the isometry can be considered as a map $F : U \rightarrow V$ with U and V two open sets of \mathbb{R}^n . Denote by x_1, \dots, x_n the coordinate functions in V . The volume form of the second manifold can be written as

$$d\text{vol}_V = \eta d\mathcal{L}^n,$$

for some C^∞ function $\eta : V \rightarrow \mathbb{R}$ bounded away from zero on compact sets.

For each $i = 1, \dots, n$, set $g_i := \eta^{-1} \nabla_H^* (\eta \nabla_H x_i)$. We have $g_i \in C^\infty(V)$ and

$$\int_V \langle \nabla_H x_i, \nabla_H v \rangle \eta \, d\mathcal{L}^n = \int_V g_i v \eta \, d\mathcal{L}^n, \quad \forall v \in \text{Lip}_c(V).$$

Notice that, for all $i = 1, \dots, n$, one has that $x_i \circ F$ and $g_i \circ F$ are Lipschitz function with respect to the control distance. Recalling Remark 2.21, we observe that as a consequence $x_i \circ F, g_i \circ F \in W_{H,\text{loc}}^{1,p}(U)$, for all $1 \leq p < \infty$. Using Proposition 3.5 with $k = 1$, we have that $F_i := x_i \circ F \in W_{H,\text{loc}}^{2,p}(U, \mathcal{L}^n)$. By Proposition 3.6, it follows that $g \circ F \in W_{H,\text{loc}}^{2,p}(U, \mathcal{L}^n)$. Iterating Proposition 3.5 and Proposition 3.6, one obtains $F_i \in W_{H,\text{loc}}^{k,p}(U, \mathcal{L}^n)$ for all $k \in \mathbb{N}$. Invoking Proposition 2.27 we finally conclude $F_i \in C^\infty(U)$, for all $i = 1, \dots, n$. \square

3.4. Proof of Theorem 1.6. Since $\text{Isom}(M)$ is locally compact by Ascoli-Arzelà's theorem, from a result of Montgomery and Zippin [MZ74, page 208, Theorem 2], we have (i). To show (ii), we fix an auxiliary Riemannian extension \tilde{g} on M and define

$$g = \int_H F_* \tilde{g} \, d\mu_H(F),$$

where μ_H is a probability Haar measure on H . Notice that $\mathcal{G} := \{F_* g : F \in H\}$ is a compact set of Riemannian tensors extending the subRiemannian structure on M . In particular, in local coordinates, all $g' \in \mathcal{G}$ can be represented with a matrix with uniformly

bounded eigenvalues. Hence, g is a Riemannian tensor, which is H -invariant, by linearity of integrals. \square

3.5. Proof of Corollary 1.8. Consider $F := g^{-1} \circ f$. So F is an isometry of M that fixes p and such that $dF|_{\Delta_p} = \text{id}_{\Delta_p}$. By Corollary 1.7, there exists a Riemannian metric \hat{g} for which F is a Riemannian isometry. Since Δ is bracket generating, one shows (see [LDO14]) that $dF|_{T_p M} = \text{id}_{T_p M}$ and hence $F = \text{id}_M$. \square

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