Turbulent flows as generalized Kelvin-Voigt materials: modeling and analysis

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Abstract

We perform a new modeling procedure for a 3D turbulent fluid, evolving toward a statistical equilibrium. This will result to add to the equations for the mean field $(\mathbf{v}, p)$ the term $-\alpha \nabla \cdot (\ell(\mathbf{x}) D\mathbf{v}_t)$, which is of the Kelvin-Voigt form, where the Prandtl mixing length $\ell = \ell(\mathbf{x})$ is not constant and vanishes at the solid walls. We get estimates for mean velocity $\mathbf{v}$ in $L_t^\infty H_x^1 \cap W_t^{1,2} H_x^{1/2}$, that allow us to prove the existence and uniqueness of a regular-weak solutions $(\mathbf{v}, p)$ to the resulting system, for a given fixed eddy viscosity. We then prove a structural compactness result that highlights the robustness of the model. This allows us to consider Reynolds averaged equations and pass to the limit in the quadratic source term in the equation for the turbulent kinetic energy k. This yields the existence of a weak solution to the corresponding Navier-Stokes turbulent kinetic energy system satisfied by $(\mathbf{v}, p, k)$.

1 Introduction

The purpose of this paper is to model incompressible turbulent flows as generalized viscoelastic materials involving the Prandtl mixing length ℓ (see e.g. Prandtl in [26]), to show the existence and uniqueness of regular-weak solutions to the resulting system of Partial Differential Equations (PDE),

$$(1.1) \quad \begin{cases} \mathbf{v}_t - \alpha \nabla \cdot (\ell(\mathbf{x}) D\mathbf{v}_t) + \nabla \cdot (\mathbf{v} \otimes \mathbf{v}) - \nu \Delta \mathbf{v} - \nabla \cdot (\nu_{\mathrm{turb}} D\mathbf{v}) + \nabla p = \mathbf{f}, \\ \nabla \cdot \mathbf{v} = 0, \end{cases}$$

for a given turbulent viscosity (eddy viscosity) ν_{turb}, when the motion takes place in a bounded smooth domain Ω and Dirichlet conditions are given at the boundary. We then

study the existence weak solutions to the corresponding NSTKE[1] system,

$$(1.2) \quad \begin{cases} \mathbf{v}_t - \alpha \nabla \cdot (\ell(\mathbf{x})D\mathbf{v}_t) + \nabla \cdot (\mathbf{v} \otimes \mathbf{v}) - \nu \Delta \mathbf{v} - \nabla \cdot (\nu_{\text{turb}}(k)D\mathbf{v}) + \nabla p = \mathbf{f}, \\ \nabla \cdot \mathbf{v} = 0, \\ k_t + \mathbf{v} \cdot \nabla k - \nabla \cdot (\mu_{\text{turb}}(k)\nabla k) = \nu_{\text{turb}}(k)|D\mathbf{v}|^2 - (\ell + \eta)^{-1}k\sqrt{|k|}, \end{cases}$$

where, to fix the notation,

- $\mathbf{v}$ is the mean velocity[2], $\mathbf{v}_t = \dfrac{\partial \mathbf{v}}{\partial t}$;

- $D\mathbf{v} = \frac{1}{2}(\nabla \mathbf{v} + \nabla \mathbf{v}^t)$ is the deformation stress;

- p is the mean pressure;

- k is the Turbulent Kinetic Energy (TKE);

- $\nu > 0$ is the kinematic viscosity, ν_{turb} the eddy viscosity;

- μ_t is the eddy diffusion and $\eta > 0$ is a small constant;

- the length scale α is that of the boundary layer, given by the relation

$$(1.3) \qquad \alpha = \frac{\nu}{u_\star},$$

  here $u_\star$ is the so called friction velocity (see [7]);

- $\mathbf{f}$ is a given source term.

As usual, the systems are set in a bounded Lipschitz domain $\Omega \subset \mathbb{R}^3$. The mixing length $\ell = \ell(\mathbf{x}) \geq 0$ is defined over Ω and, according to well known physical laws (see (2.5) and (2.6) below), $0 \leq \ell \in C^1(\overline{\Omega})$ and vanishes at the boundary $\Gamma = \partial\Omega$ as follows:

$$(1.4) \qquad \ell(\mathbf{x}) \simeq d(\mathbf{x}, \Gamma) = \rho(\mathbf{x}), \quad \text{when} \quad \mathbf{x} \to \Gamma, \, \mathbf{x} \in \Omega,$$

where $d(\mathbf{x}, \Gamma)$ denotes the distance of the point $\mathbf{x}$ from the boundary.
Model (1.1) is close to viscoelastic materials models, given by the Kelvin-Voigt relation:

$$(1.5) \qquad \boldsymbol{\sigma} = E\,\boldsymbol{\varepsilon} + \eta\,\boldsymbol{\varepsilon}_t,$$

where $\boldsymbol{\sigma}$ denotes the Cauchy stress tensor and $\boldsymbol{\varepsilon}$ the strain-rate tensor. In this case, E is the modulus of elasticity and η the viscosity (see for instance Germain [10] or Gurtin [12]). In fluid mechanics, $\boldsymbol{\varepsilon} = D\mathbf{v}$, and this model is used to describe some non Newtonian fluids, such as lubricants. For such flows, the law (1.5) becomes

$$\boldsymbol{\sigma} = -p\,\text{Id} + \nu D\mathbf{v} + \gamma^2 D\mathbf{v}_t \qquad \nu, \gamma \in \mathbb{R}^+,$$

that yields the incompressible Navier-Stokes-Voigt equations:

$$(1.6) \qquad \begin{cases} \mathbf{v}_t - \gamma^2 \Delta \mathbf{v}_t + \nabla \cdot (\mathbf{v} \otimes \mathbf{v}) - \nu \Delta \mathbf{v} + \nabla p = \mathbf{f}, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \nabla \cdot \mathbf{v} = 0. \end{cases}$$

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[1]Here, RANS = Reynolds Averaged Navier-Stokes and NSTKE = Navier-Stokes-Turbulent-Kinetic-Energy. NSTKE model is a specific RANS model.

[2]Usually, the mean (or averaged) velocity is denoted by $\overline{\mathbf{v}}$. Throughout the paper we omit the over-line for simplicity, except in Section 2, devoted to turbulence modelling.

Mathematical investigations about system (1.6) were first carried out by Oskolkov [24, 25], who proved the existence and uniqueness of weak and strong solutions in some particular sense.. Then, several mathematical problems concerning (1.6) have been studied by Titi *et al.* [13, 17, 27], making a clear relation between Navier-Stokes-Voigt and turbulence modeling. In addition, in [14] Larios & Titi showed the connection between the Navier-Stokes-Voigt equations and the simplified Bardina's model introduced in [15], designed as a Large-Eddy simulation model. In addition an interpretation of the Navier-Stokes-Voigt equations in terms of approximate deconvolution models is also given in [5].

In this paper we connect the Prandtl-Smagorinsky's model with the Turbulent Kinetic Energy (TKE) model, to calculate the eddy viscosity ν_{turb}. To make it clear, let $\boldsymbol{\sigma}^{(\text{R})}$ denotes the Reynolds stress. We will show how, combining the energy inequality with the equation satisfied by k (without any closure assumption), we are led to set –in certain specific regimes, such as the convergence to stable statistical states see (2.13)– the following constitutive law

$$\boldsymbol{\sigma}^{(\text{R})} = -\alpha\ell\, D\mathbf{v}_t - \nu_{\text{turb}} D\mathbf{v} + \frac{2}{3}k\,\text{Id},$$

instead of the usual one

$$\boldsymbol{\sigma}^{(\text{R})} = -\nu_{\text{turb}} D\mathbf{v} + \frac{2}{3}k\,\text{Id},$$

associated with the classical Boussinesq assumption. This yields the PDE system (1.1) including the term $-\alpha\nabla\cdot(\ell D\mathbf{v}_t)$, and then also the NSTKE system (1.2) after having performed the usual closure procedure about k, where $\nu_{\text{turb}} = \nu_{\text{turb}}(k) = \ell\sqrt{k}$.

Turning to the analysis of the systems, we observe that –according to assumption (1.4) about the mixing length ℓ– the additional generalized Kelvin-Voigt term $-\alpha\nabla\cdot(\ell D\mathbf{v}_t)$ enforces for the equations a natural functional structure in $H^{1/2}(\Omega) = [H^1(\Omega), L^2(\Omega)]_{1/2}$, cf. Lions & Magenes [21], which is a critical scaling-invariant space for the Navier-Stokes equations. In particular, we obtain for the velocity sharp estimates in $W^{1,2}(0,T;H^{1/2}(\Omega)^3)$, as well as in $L^\infty(0,T;H_0^1(\Omega)^3)$. We are then able to prove the existence and uniqueness of regular-weak solution to (1.1) (see Theorem 4.1 and the generalisation in Theorem 4.2).

However, we believe that the most interesting result of this paper is the compactness result we prove in Lemma 5.1. We consider an eddy viscosities sequence $(\nu_{\text{turb}}^n)_{N\in\mathbf{N}}$ which is bounded in $L^\infty([0,\infty[\times\Omega)$ and in addition converges *a.e.* to ν_{turb} in $[0,\infty[\times\Omega$ as $n\to\infty$. We also show that the corresponding regular-weak sequence of solution $(\mathbf{v}^n)_{N\in\mathbf{N}}$ converges, in some sense, to the regular-weak solution $\mathbf{v}$ of the limit problem with ν_{turb} as eddy viscosity. Moreover, we get the convergence of the energies, that is $\nu_{\text{turb}}^n|D\mathbf{v}^n|^2 \to \nu_{\text{turb}}|D\mathbf{v}|^2$ in the sense of the measures.

This compactness result allows us to prove the existence of a solution to the NSTKE-Voigt system (1.2) (see Theorem 5.1 below). We stress that the usual system coupling $\mathbf{v}$, p. and k only yields a variational inequality for k when passing to the limit in the equations, because of the lack of strong convergence of the energies (see [7, 18]). This observation makes Theorem 5.1 a relevant and original result.

Plan of the paper. The paper is organized as follows: Section 2 is devoted to modeling and to explain the motivations for the systems of PDE we study. Then, in Section 3 we use functional analysis and interpolation theory to provide estimates in various spaces, especially in $H^{1/2}(\Omega)$. The proof of the existence and uniqueness results for the generalized Navier-Stokes-Voigt equations (4.1) and then also to the model (1.1) is developed in Section 4. Finally, the compactness result and analysis of the NSTKE-Voigt system is performed in Section 5.

Acknowledgments

2 Kelvin-Voigt modeling for turbulent flows

In this section (and only in this section) $\mathbf{v}$ and p denote the velocity and pressure of the fluid respectively (and not the mean fields unlike in the rest of the paper). Hence, the couple $(\mathbf{v}, p)$ solves the Navier-Stokes equations (NSE),

$$(2.1) \qquad \begin{cases} \mathbf{v}_t + \nabla \cdot (\mathbf{v} \otimes \mathbf{v}) - \nu \Delta \mathbf{v} + \nabla p = \mathbf{f}, \\ \nabla \cdot \mathbf{v} = 0. \end{cases}$$

We first recall some results about basic turbulence modeling to derive the equation for the mean $\overline{\mathbf{v}}$ and to define the Reynolds stress $\boldsymbol{\sigma}^{(\mathrm{R})}$. Then, we show how –when simultaneously using the Prandtl-Smagorinsky, the turbulent kinetic energy models, and the equation satisfied by the TKE– we get the additional term $-\alpha \nabla \cdot (\ell\, D\overline{\mathbf{v}}_t)$ in the equation for $\overline{\mathbf{v}}$. This occurs in specific regimes, such as the convergence to a statistical equilibrium (see Remark 2.2).

We wish to mention that a very close modeling process has been previously performed in Rong, Layton, and Zhao [28]. The latter paper gave to us some inspiration for the modelling procedure we develop here. One main difference is that we study the TKE equations, while in their paper, Rong, Layton, and Zhao considered a rotational structure, without involving the equation for the turbulent kinetic energy. Moreover, they were looking at back-scatter terms, so that our point of view and interpretation are –at the very end– rather different.

2.1 Recalls of basic turbulence modeling

According to the Reynolds decomposition, $\mathbf{v}$ and p are decomposed as the sum of their mean and fluctuation (cf. [3, 7])

$$\mathbf{v} = \overline{\mathbf{v}} + \mathbf{v}', \quad \text{and} \quad p = \overline{p} + p',$$

where the averaging filter is linear, commutes with any differential operator (namely $D\overline{\psi} = \overline{D\psi}$), and it is idempotent (that is $\overline{\overline{\psi}} = \overline{\psi}$). From these assumptions, one gets the relation

$$\overline{\mathbf{v} \otimes \mathbf{v}} = \overline{\mathbf{v}} \otimes \overline{\mathbf{v}} + \boldsymbol{\sigma}^{(\mathrm{R})},$$

where the Reynolds stress $\boldsymbol{\sigma}^{(\mathrm{R})}$ is given by

$$\boldsymbol{\sigma}^{(\mathrm{R})} := \overline{\mathbf{v}' \otimes \mathbf{v}'}.$$

Therefore, applying the mean operator to the NSE (2.1) yields

$$(2.2) \qquad \begin{cases} \overline{\mathbf{v}}_t + \nabla \cdot (\overline{\mathbf{v}} \otimes \overline{\mathbf{v}}) - \nu \Delta \overline{\mathbf{v}} + \nabla \cdot \boldsymbol{\sigma}^{(\mathrm{R})} + \nabla \overline{p} = \overline{\mathbf{f}}, \\ \nabla \cdot \overline{\mathbf{v}} = 0. \end{cases}$$

To "close" (2.2), one must express $\boldsymbol{\sigma}^{(\mathrm{R})}$ in terms of mean quantities. As we already said in the introduction, the Boussinesq assumption [6] yields

$$\boldsymbol{\sigma}^{(\mathrm{R})} = -\nu_{\mathrm{turb}} D\overline{\mathbf{v}} + \frac{2}{3}k\,\mathrm{Id},$$

where we recall that ν_{turb} is the eddy viscosity, $k = \frac{1}{2}\overline{|\mathbf{v}'|^2}$ the turbulent kinetic energy (TKE), and $D\mathbf{v} = \frac{1}{2}(\nabla\mathbf{v} + \nabla\mathbf{v}^T)$ the deformation tensor.

One main challenge in turbulence modelling is the determination of ν_{turb}. In what follows, we combine the Prandtl-Smagorinsky's model

$$(2.3) \qquad\qquad \nu_{\mathrm{turb}} = \ell\sqrt{\alpha\ell}|D\overline{\mathbf{v}}|,$$

where the boundary layer length scale α is given by (1.3), and the NSTKE model with ν_{turb} is given by

$$(2.4) \qquad\qquad \nu_{\mathrm{turb}} = \ell\sqrt{k}.$$

Dimensionless constants may be involved in the above equations. We have set them equal to 1 for the sake of simplicity.

Observe that both models involve the Prandtl mixing length ℓ. In the case of a flow over a plate $\Omega = \mathbb{R}^2 \times \{z > 0\}$, one finds in Obukhov [23] the following law

$$(2.5) \qquad\qquad \ell = \ell(z) = \kappa z,$$

where $\kappa \in [0.35, 0.42]$ is the Von Kármán constant. A more sophisticated formula (however very popular especially for the use in the computation of the turbulent channel flow) can be found in Van Driest [30]:

$$(2.6) \qquad\qquad \ell = \ell(z) = \kappa z(1 - e^{-z/A}),$$

where A depends on the oscillations of the plate and ν. Alternative formulas are provided in [20]. In all cases, ℓ satisfies the law (1.4) and in particular it vanishes at the solid boundary of the domain.

2.2 Modelling process

We start from the natural energy inequality[3] deduced from the equation (2.2) by integration by parts, at any time positive time t (assuming as usual that solutions are smooth enough to carry on all the calculations)

$$(2.7) \qquad\qquad \langle \overline{\mathbf{v}}_t, \overline{\mathbf{v}}\rangle + \nu\|\nabla\overline{\mathbf{v}}(t)\|^2 + \langle\nabla \cdot \boldsymbol{\sigma}^{(\mathrm{R})}, \overline{\mathbf{v}}(t)\rangle \leq \langle\overline{\mathbf{f}}(t), \overline{\mathbf{v}}(t)\rangle.$$

We aim to formulate the contribution of the term

$$\mathscr{T}(t) = \langle\nabla \cdot \boldsymbol{\sigma}^{(\mathrm{R})}, \overline{\mathbf{v}}(t)\rangle,$$

by means of mean quantities. This which will be deduced by using the equation satisfied by k (see [7, Sec. 4.4.1])

$$\partial_t k + \nabla \cdot (\overline{\mathbf{v}}\,k + \overline{e'\mathbf{v}'}) = -\boldsymbol{\sigma}^{(\mathrm{R})} : \nabla\overline{\mathbf{v}} - \varepsilon + \overline{\mathbf{f}' \cdot \mathbf{v}'},$$

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[3]We use $\|\cdot\|$ for the L^2-norm in this section and $\langle\,.\,,\,.\,\rangle$ for the associated scalar product.

where $e = k + e' = \frac{1}{2}|\mathbf{v}'|^2$ denotes the kinetic energy of the fluctuations, and ε is the turbulent dissipation,

$$\varepsilon := \nu\overline{|D\mathbf{v}'|^2}.$$

Integrating formally the equation satisfied by k in space –leaving apart eventual boundary condition issues– leads to

$$\frac{d}{dt}\int_\Omega k(t) = \mathscr{T}(t) - \int_\Omega \varepsilon(t) + \overline{\langle \mathbf{f}', \mathbf{v}'\rangle},$$

that we insert in the inequality (2.7) to obtain

$$(2.8) \qquad \langle \overline{\mathbf{v}}_t, \overline{\mathbf{v}}\rangle + \frac{d}{dt}\int_\Omega k + \nu\|\nabla\overline{\mathbf{v}}(t)\|^2 + \int_\Omega \varepsilon(t) \leq \langle \overline{\mathbf{f}}(t), \overline{\mathbf{v}}(t)\rangle + \overline{\langle \mathbf{f}'(t), \mathbf{v}'(t)\rangle}.$$

In order to eliminate the term $\dfrac{d}{dt}\displaystyle\int_\Omega k$ from (2.8), we enforce equality between the Prandtl-Smagorinsky's model (2.3) and the NSTKE one (2.4), which leads to the closure equality

$$(2.9) \qquad k = \alpha\,\ell|D\overline{\mathbf{v}}|^2.$$

Then, by using (2.9), we get the formal identity

$$(2.10) \qquad \frac{d}{dt}\int_\Omega k = \alpha\frac{d}{dt}\int_\Omega \ell|D\overline{\mathbf{v}}|^2 = -2\langle \alpha\nabla\cdot(\ell D\overline{\mathbf{v}}_t), \overline{\mathbf{v}}\rangle.$$

Finally, we combine (2.8) with (2.10), which leads to the inequality

$$(2.11) \qquad \langle \overline{\mathbf{v}}_t - 2\alpha\nabla\cdot(\ell D\overline{\mathbf{v}}_t) - \nu\Delta\overline{\mathbf{v}}, \overline{\mathbf{v}}\rangle + \int_\Omega \varepsilon(t) \leq \langle \overline{\mathbf{f}}(t), \overline{\mathbf{v}}(t)\rangle + \overline{\langle \mathbf{f}'(t), \mathbf{v}'(t)\rangle}.$$

Comparing (2.7) with (2.11), suggests to put (up to a redefinition of the parameter α)

$$\boldsymbol{\sigma}^{(\mathrm{R})} = -\alpha\ell D\overline{\mathbf{v}}_t - \nu_{\mathrm{turb}}D\overline{\mathbf{v}} + \frac{2}{3}k\,\mathrm{Id},$$

and yields the following energy inequality

$$(2.12) \qquad \frac{1}{2}\frac{d}{dt}(\|\overline{\mathbf{v}}(t)\|^2 + \alpha\|\sqrt{\ell}D\overline{\mathbf{v}}\|^2) + \nu\|\nabla\overline{\mathbf{v}}(t)\|^2 + \|\sqrt{\nu_{\mathrm{turb}}}D\overline{\mathbf{v}}\|^2 \leq \langle \overline{\mathbf{f}}(t), \overline{\mathbf{v}}(t)\rangle.$$

Comparing inequalities (2.12) and (2.11), we see that all this makes sense when:

$$(2.13) \qquad \|\sqrt{\nu_{\mathrm{turb}}}D\overline{\mathbf{v}}\|^2 + \overline{\langle \mathbf{f}'(t), \mathbf{v}'(t)\rangle} \leq \|\sqrt{\varepsilon(t)}\|^2,$$

and in this case the system satisfied by $\overline{\mathbf{v}}$ becomes

$$\begin{cases} \overline{\mathbf{v}}_t - \alpha\nabla\cdot(\ell D\overline{\mathbf{v}}_t) + \nabla\cdot(\overline{\mathbf{v}}\otimes\overline{\mathbf{v}}) - \nu\Delta\overline{\mathbf{v}} - \nabla\cdot(\nu_{\mathrm{turb}}D\overline{\mathbf{v}}) + \nabla\left(\overline{p} + \frac{2}{3}k\right) = \overline{\mathbf{f}}, \\[2mm] \nabla\cdot\overline{\mathbf{v}} = 0. \end{cases}$$

Remark 2.1. *When $\ell(\mathbf{x})$ is constant and equal to α (to set ideas), and as $\nabla\cdot\overline{\mathbf{v}}_t = 0$, we have $\alpha\nabla\cdot(\ell(\mathbf{x})D\overline{\mathbf{v}}_t) = \alpha^2\Delta\overline{\mathbf{v}}_t$. Therefore, we get in this case the usual (constant coefficients) Kelvin-Voigt term involed in Equation (1.6). Observe that in our model the linear differential operator $-\nabla\cdot(\ell(\mathbf{x})\,D)$ turns out to be degenerate at the boundary, hence different mathematical tools need to the invoked.*

Remark 2.2. *Condition* (2.13) *asks for some comments. To see if it can be justified, let us take a constant source term* $\mathbf{f}(t) = \mathbf{f}$, *without turbulent fluctuation, which means* $\mathbf{f}' = \mathbf{0}$. *In this case relation* (2.13) *simplifies to*

$$(2.14) \qquad \|\sqrt{\nu_{\mathrm{turb}}}D\overline{\mathbf{v}}\|^2 \leq \|\sqrt{\varepsilon(t)}\|^2.$$

The usual closed equation for k *is*

$$k_t + \overline{\mathbf{v}} \cdot \nabla k - \nabla \cdot (\mu_{\mathrm{turb}}\nabla k) = \nu_{\mathrm{turb}}|D\overline{\mathbf{v}}|^2 - \varepsilon,$$

which gives (while ignoring possible boundary conditions)

$$\frac{d}{dt}\int_\Omega k = \|\sqrt{\nu_{\mathrm{turb}}}D\overline{\mathbf{v}}\|^2 - \|\sqrt{\varepsilon(t)}\|^2.$$

Therefore, (2.14) *indicates a decrease of TKE, which means a decrease of the turbulence, towards a laminar state, or a stable statistical equilibrium, such as a grid turbulence.*

3 Functional setting and estimate

The analysis of system (1.1) yields immediately standard a priori estimates in $L_t^\infty L_x^2$ and $L_t^2 H_x^1$, taking the solution itself as test function. The question is whether the Voigt term $-\alpha\nabla \cdot (\ell D\mathbf{v}_t)$ provides additional regularity as in the case with $0 < l = \mathrm{const.}$, see Larios and Titi [14]. The issue is the degeneration of the mixing length ℓ at the boundary, according to (3.1) below. The purpose of this section is to derive from the interpolation theory a general estimate, that will later on enable us to show that the term $-\alpha\nabla \cdot (\ell D\mathbf{v}_t)$ yields additional $W^{1,2}(0, T; H^{1/2})$ and $L^\infty(0, T; H^1)$ regularity.

3.1 Framework and preliminaries

As usual in mathematical fluid dynamics, we use the following spaces,

$$\begin{aligned}
\mathcal{V} &= \left\{\boldsymbol{\varphi} \in \mathcal{D}(\Omega)^3, \ \nabla \cdot \boldsymbol{\varphi} = 0 \ \text{in} \ \Omega\right\}, \\
H &= \left\{\mathbf{v} \in L^2(\Omega)^3, \ \nabla \cdot \mathbf{v} = 0 \ \text{in} \ \Omega, \ \mathbf{v} \cdot \mathbf{n} = 0 \ \text{on} \ \Gamma\right\}, \\
V &= \left\{\mathbf{v} \in H_0^1(\Omega)^3, \ \nabla \cdot \mathbf{v} = 0 \ \text{in} \ \Omega\right\},
\end{aligned}$$

where $\mathcal{D}(\Omega) = C_0^\infty(\Omega)$ with the topology used in distribution theory and we recall that $\mathcal{V}$ is dense in H and V for their respective topologies, see Girault and Raviart [11]. Here $L^2(\Omega)$ and $H_0^1(\Omega)$ stand for the usual Lebesgue and Sobolev spaces.

Throughout the rest of the paper, the mixing length $\ell = \ell(\mathbf{x}) \in C^1(\overline{\Omega})$ is such that

$$(3.1) \qquad \begin{cases} \forall\, K \subset \Omega, \quad K \ \text{compact}, \quad \inf_K \ell > 0, \\ \ell(\mathbf{x}) \simeq d(\mathbf{x}, \Gamma) = \rho(\mathbf{x}), \quad \text{when} \quad \mathbf{x} \to \Gamma, \ \text{for} \ \mathbf{x} \in \Omega. \end{cases}$$

According to the classical interpolation theory we recall that

$$H^{1/2}(\Omega) = [H^1(\Omega), L^2(\Omega)]_{1/2},$$

and also, recalling the behavior of ℓ at the boundary from (3.1), we can introduce the Lions-Magenes space

$$H_{00}^{1/2}(\Omega) = [H_0^1(\Omega), L^2(\Omega)]_{1/2} = \left\{u \in H^{1/2}(\Omega), \ \text{s.t.} \ \ell^{-1/2}u \in L^2(\Omega)\right\},$$

cf. [21, Ch. 1]. In the following we will consider the following Hilbert space

$$V_{1/2} = \left\{ \mathbf{v} \in H^{1/2}(\Omega)^3 \,;\; \nabla \cdot \mathbf{v} = 0 \text{ in } \Omega \quad \text{and} \quad \mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \Gamma \right\}$$

equipped with the norm of $H^{1/2}(\Omega)^3$.

Finally, recall that when Ω is connected, the differential operator $D = \frac{\nabla + \nabla^t}{2}$ is well defined over $H^s(\Omega)^3$ whatever $s \geq 0$; In addition (see Nečas and I. Hlaváček [22]),

$$\mathrm{K} := \mathrm{Ker}\, D = \left\{ \mathbf{v} \in H^s(\Omega)^3 \text{ s.t. } \exists\, (\mathbf{a}, \mathbf{b}) \in \mathbb{R}^3 \times \mathbb{R}^3;\; \mathbf{v}(\mathbf{x}) = \mathbf{b} \times \mathbf{x} + \mathbf{a} \right\},$$

and we recall the following Korn inequality

$$(3.2) \qquad \exists\, c(\Omega) : \; \forall\, \mathbf{v} \in H^1(\Omega)^3, \quad \|\mathbf{v}\|_{H^1(\Omega)^3/\mathrm{K}} \leq C \|D\mathbf{v}\|_{L^2(\Omega)^9},$$

where for any given Banach space B and any closed subspace $E \subset B$, B/E denotes the quotient space. Moreover, for any $\mathbf{v} \in H_0^1(\Omega)^3$, we have $\|\mathbf{v}\|_{H^1(\Omega)^3} \leq C \|D\mathbf{v}\|_{L^2(\Omega)^9}$, because in this case the kernel K is reduced to $\mathbf{0}$.

3.2 Main general estimate

We deduce now the most relevant inequality, which will be used to prove a priori estimates for the generalized Voigt model, when using the solution itself as test function.

Theorem 3.1. *Let $\mathbf{v} \in \mathcal{D}'(\Omega)^3$ be such that $\sqrt{\ell} D\mathbf{v} \in L^2(\Omega)^9$. Then $\mathbf{v} \in H^{1/2}(\Omega)^3$ and there exists a constant $C = C(\Omega)$ such that*

$$(3.3) \qquad \|\mathbf{v}\|_{H^{1/2}(\Omega)^3/\mathrm{K}} \leq C \|\sqrt{\ell} D\mathbf{v}\|_{L^2(\Omega)^9}.$$

In particular,

$$(3.4) \qquad W = \left\{ \mathbf{v} \in H;\; \sqrt{\ell} D\mathbf{v} \in L^2(\Omega)^9 \right\} \hookrightarrow V_{1/2},$$

with continuous embedding.

Proof. We argue in two steps.

Step 1. Let $\mathbf{v} \in \mathcal{D}'(\Omega)^3$ such that $\sqrt{\ell} D\mathbf{v} \in L^2(\Omega)^9$ and $\boldsymbol{\varphi} \in \mathcal{D}(\Omega)^9$. As $D\mathbf{v} \in L^2_{\mathrm{loc}}(\Omega)^3$, then we have

$$|\langle D\mathbf{v}, \boldsymbol{\varphi}\rangle| = \left| \int_\Omega \sqrt{\ell}\, D\mathbf{v} : \frac{\boldsymbol{\varphi}}{\sqrt{\ell}} \right| \leq C \|\sqrt{\ell}\, D\mathbf{v}\|_{L^2(\Omega)^9} \|\boldsymbol{\varphi}\|_{H_{00}^{1/2}(\Omega)^9}.$$

Because of the density of $\mathcal{D}(\Omega)$ in $H_{00}^{1/2}(\Omega)$, this shows that $D\mathbf{v} \in \left[H_{00}^{1/2}(\Omega)^9 \right]'$ with the estimate

$$(3.5) \qquad \|D\mathbf{v}\|_{\left[H_{00}^{1/2}(\Omega)^9 \right]'} \leq C \|\sqrt{\ell}\, D\mathbf{v}\|_{L^2(\Omega)^9}.$$

Step 2. According to [1, 8], we have

$$\forall\, \mathbf{v} \in L^2(\Omega)^3, \quad \|\mathbf{v}\|_{L^2(\Omega)^3/\mathrm{K}} \leq C \|D\mathbf{v}\|_{H^{-1}(\Omega)^9}.$$

Therefore, we deduce from classical interpolation theorems and from the following identities (see in [21]),

$$[H^1(\Omega)^3/\mathrm{K}, L^2(\Omega)^3/\mathrm{K}]_{1/2} = H^{1/2}(\Omega)^3/\mathrm{K}, \quad \text{and} \quad [L^2(\Omega), H^{-1}(\Omega)]_{1/2} = [H_{00}^{1/2}(\Omega)]',$$

the inequality

$$\|\mathbf{v}\|_{H^{1/2}(\Omega)^3/\mathrm{K}} \leq C \|D\mathbf{v}\|_{\left[H_{00}^{1/2}(\Omega)^9 \right]'}.$$

Hence the estimate (3.3) follows by using (3.5) and obviously also the embedding (3.4). □

4 Well-posedness for the generalized Navier-Stokes-Voigt equations

In this section we start with the analysis of system (1.1) without eddy viscosity, that means $\nu_{\mathrm{turb}} = 0$. This is done both for simplicity of presentation and to highlight the role of the generalized Voigt term. The resulting system, called *generalized Navier-Stokes-Voigt equations*, is the following:

(4.1)
$$
\begin{cases}
\mathbf{v}_t - \alpha \nabla \cdot (\ell \, D\mathbf{v}_t) + (\mathbf{v} \cdot \nabla)\,\mathbf{v} - \nu \Delta \mathbf{v} + \nabla p = \mathbf{f} & \text{in } (0,T) \times \Omega, \\
\nabla \cdot \mathbf{v} = 0 & \text{in } (0,T) \times \Omega, \\
\mathbf{v}_{|\Gamma} = 0 & \text{on } (0,T) \times \Gamma, \\
\mathbf{v}_{|t=0} = \mathbf{v}_0 & \text{in } \Omega,
\end{cases}
$$

which is set in $Q_T = (0,T) \times \Omega$, where Ω is a given Lipschitz bounded domain in $\mathbb{R}^3$ with its boundary $\Gamma = \partial \Omega$, T a fixed positive time[4], and ℓ satisfies (1.4). The main results of this section are the existence and uniqueness of regular-weak solutions (see Definition 4.1 below), when the initial velocity $\mathbf{v}_0 \in V$.

Throughout the rest of the paper, the L^2-norm of a given u is simply denoted by $\|u\|$, while $\|\cdot\|_p$ and $\|\cdot\|_{s,p}$ denote the standard $L^p(\Omega)$ and $W^{s,p}(\Omega)$ norms, respectively.

4.1 Strong solutions

This aim of this subsection is to prove that given a positive finite time T, any strong (classical) solution $\mathbf{v}$ of (4.1) has natural bounds in $L^\infty(0,T;V) \cap W^{1,2}(0,T;V_{1/2})$ derived from energy balances, showing that the term $-\alpha \nabla \cdot (\ell(\mathbf{x})\,D\mathbf{v}_t)$ –despite the degeneracy at the boundary– brings a strong regularizing effect on the system. In particular, the generalized Voigt term provides stronger a priori estimate when compared to the usual (non regularized) Navier-Stokes equations, since it allows to show bounds in critical scaling-invariant spaces *à la* Kato-Fujita. These estimates are essential for proving the existence result of the next subsection.

Following [16, 19], when considering $\mathbf{v}_0 \in V \cap C(\overline{\Omega})^3$, we say that $(\mathbf{v}, p)$ is a strong solution to (4.1) over $Q_T = [0,T] \times \Omega$, if

- $\forall \tau < T$, $\mathbf{v} \in C^2(Q_\tau)^3$, $p \in C^1(Q_\tau)$, and they satisfy the relations ((4.1), i), ii)) in the classical sense in $Q_\tau = [0,\tau] \times \Omega$,

- $\mathbf{v}(t,\cdot) \in C(\overline{\Omega})^3$ for all $t < T$, and $\mathbf{v}(t,\cdot) = 0$ on Γ,

- $\mathbf{v}(t,\cdot)$ uniformly converges to $\mathbf{v}_0$ as $t \to 0^+$.

Remark 4.1. *We frequently talk about the velocity $\mathbf{v}$ as a strong solution, without mentioning the pressure p. This means that we have implicitly projected the system over divergence-free vector fields by the Leray projector, which eliminates the pressure. The pressure can be recovered via the De Rham procedure (see e.g. Temam [29]).*

Remark 4.2. *We say that a strong solution $\mathbf{v}$ of (4.1) has a singularity at a given time $0 < T^\star < \infty$ if $\|\mathbf{v}(t)\|_\infty \to \infty$ as $t \to T^\star$, $t < T^\star$. At this stage, we are not able to show that any strong solution has no singularity. We do not even know if there exist strong solutions, which remains an open problem.*

---

[4]Remind that when $\nabla \cdot \mathbf{v} = 0$, then $\nabla \cdot (\mathbf{v} \otimes \mathbf{v}) = (\mathbf{v} \cdot \nabla)\,\mathbf{v}$. We use either of these forms without necessarily warning, depending on the situation.

The estimates we get are based on the following non standard version of Gronwall's Lemma, the proof of which is carried out, *e.g.*, in Emmrich [9].

Lemma 4.1. *Let* $\lambda \in L^1(0,T)$, *with* $\lambda(t) \geq 0$ *for almost all* $t \in [0,T]$, *let* $g \in C([0,T])$ *be a non-decreasing function, and let* $f \in L^\infty(0,T)$, *such that* $\forall\, t \in [0,T]$, *it holds*

$$f(t) \leq g(t) + \int_0^t \lambda(s) f(s)\, ds.$$

Then, we have

$$f(t) \leq g(t) \exp\left(\int_0^t \lambda(s)\, ds\right).$$

In this subsection we assume that $\mathbf{f}(t) = \mathbf{f} \in C(\overline{\Omega})$ does not depend on t, and we denote by F either $\|\mathbf{f}\|_{-1,2}^2$ or $\|\mathbf{f}\|^2$ so far non risk of confusion occurs, and C denotes any constant (normally $C_P\|\mathbf{f}\| \leq \|\mathbf{f}\|_{-1,2}$, C_P being the Poincaré's constant). Among many choices for the functional spaces of the source term (see also the discussion in the next subsection, where different choices are considered), this one has the advantage that it yields a clear and neat bound of the growth of the r.h.s in the estimates for statistical equilibrium.

The main result of this subsection is the following.

Lemma 4.2. *Let* $\mathbf{v}$ *be a strong solution of* (4.1) *in* $Q_T = [0,T] \times \Omega$, *for a* $T > 0$. *Then, the following estimates hold true for all* $s \in [0,T[$:

$$(4.2) \qquad \|\mathbf{v}(s)\|_{1/2,2}^2 + \nu \int_0^s \|\nabla \mathbf{v}(t)\|^2\, dt \leq C\left(\frac{Fs}{\nu} + E(0)(\alpha,\ell)\right),$$

$$(4.3) \qquad \nu\|\nabla \mathbf{v}(s)\|^2 \leq \left(\nu\|\nabla \mathbf{v}_0\|^2 + Fs\right) \exp\left\{\frac{C}{\alpha\nu^2}\left(\frac{Fs}{\nu} + E(0)(\alpha,\ell)\right)\right\},$$

and

$$(4.4) \quad C\alpha \int_0^s \|\mathbf{v}_t(t)\|_{1/2,2}^2\, dt + \alpha \int_0^s \|\sqrt{\ell} D\mathbf{v}_t(t)\|^2\, dt \leq Fs + \nu\|\nabla \mathbf{v}_0\|^2$$
$$+ \frac{C}{\alpha\nu^2}\left(\nu\|\nabla \mathbf{v}_0\|^2 + FT\right)\left(\frac{Fs}{\nu} + E(0)(\alpha,\ell)\right)\exp\left\{\frac{C}{\alpha\nu^2}\left(\frac{FT}{\nu} + E(0)(\alpha,\ell)\right)\right\},$$

where $2E(0)(\alpha,\ell) = \|\mathbf{v}_0\|^2 + \alpha\|\sqrt{\ell} D\mathbf{v}_0\|^2$. *In particular,* $\mathbf{v}$ *has natural bounds in* $L^\infty(0,T;V) \cap W^{1,2}(0,T;V_{1/2})$ *and* $\sqrt{\ell} D\mathbf{v}_t \in L^2(0,T;L^2(\Omega)^9)$.

Proof. We take the dot product of $((4.1),\, \mathrm{i}))$ by $\mathbf{v}$. We integrate by parts and we use the identity $\langle (\mathbf{v} \cdot \nabla)\mathbf{v}, \mathbf{v} \rangle = 0$. These calculations are justified because $\mathbf{v}$ is a strong solution, and this gives for all $t \in [0,T]$,

$$\frac{1}{2}\frac{d}{dt}(\|\mathbf{v}(t)\|^2 + \alpha\|\sqrt{\ell} D\mathbf{v}(t)\|^2) + \nu\|\nabla \mathbf{v}(t)\|^2 = \langle \mathbf{f}(t), \mathbf{v}(t)\rangle \leq \frac{F}{2\nu} + \frac{\nu}{2}\|\nabla \mathbf{v}(t)\|^2,$$

hence (4.2) follows after integration in time by using (3.3), the fact that the norm of $V_{1/2}$ is that inherited from $H^{1/2}(\Omega)^3$, and $H^{1/2}(\Omega) \hookrightarrow L^2(\Omega)$ with continuous dense injection. We next take the dot product of $((4.1),\, \mathrm{i}))$ by $\mathbf{v}_t$. In this case the non-linear term brings a contribution (in this new energy budget) given by

$$\|\mathbf{v}_t(t)\|^2 + \alpha\|\sqrt{\ell} D\mathbf{v}_t(t)\|^2 + \frac{\nu}{2}\frac{d}{dt}\|\nabla \mathbf{v}(t)\|^2 = \langle \mathbf{f}(t), \mathbf{v}_t(t)\rangle - \langle (\mathbf{v} \cdot \nabla)\mathbf{v}, \mathbf{v}_t\rangle(t),$$

As we can estimate

$$|\langle \mathbf{f}(t), \mathbf{v}_t(t)\rangle| \leq \frac{F}{2} + \frac{1}{2}\|\mathbf{v}_t(t)\|^2,$$

we obtain by using (3.3), keeping half of the contribution of the term $\alpha\|\sqrt{\ell}D\mathbf{v}_t(t)\|^2$,

$$(4.5)\quad \frac{1}{2}\|\mathbf{v}_t(t)\|^2 + C\alpha\|\mathbf{v}_t(t)\|^2_{1/2,2} + \frac{\alpha}{2}\|\sqrt{\ell}D\mathbf{v}_t(t)\|^2 + \frac{\nu}{2}\frac{d}{dt}\|\nabla\mathbf{v}(t)\|^2 \leq \frac{F}{2} + |\langle (\mathbf{v}\cdot\nabla)\,\mathbf{v}, \mathbf{v}_t\rangle(t)|.$$

To deal with the nonlinear term, we use standard interpolation inequalities. The key of the process is the continuous Sobolev embedding $H^{1/2}(\Omega) \hookrightarrow L^3(\Omega)^3$, which is the limit case. Therefore, we have

$$
\begin{aligned}
|\langle (\mathbf{v}\cdot\nabla)\,\mathbf{v}, \mathbf{v}_t\rangle(t)| &\leq \|\mathbf{v}(t)\|_6 \|\nabla\mathbf{v}(t)\|\,\|\mathbf{v}_t(t)\|_3 \\
&\leq C\|\nabla\mathbf{v}(t)\|^2\,\|\mathbf{v}_t(t)\|_{1/2,2} \\
&\leq \frac{1}{2C\alpha}\|\nabla\mathbf{v}(t)\|^4 + \frac{C\alpha}{2}\|\mathbf{v}_t(t)\|^2_{1/2,2},
\end{aligned}
$$

so that (4.5) becomes

$$(4.6)\quad \|\mathbf{v}_t(t)\|^2 + C\alpha\|\mathbf{v}_t(t)\|^2_{1/2,2} + \alpha\|\sqrt{\ell}D\mathbf{v}_t(t)\|^2 + \nu\frac{d}{dt}\|\nabla\mathbf{v}(t)\|^2 \leq F + \frac{1}{C\alpha}\|\nabla\mathbf{v}(t)\|^4.$$

In particular it follows from the above estimate that

$$\nu\frac{d}{dt}\|\nabla\mathbf{v}(t)\|^2 \leq F + \frac{1}{C\alpha}\|\nabla\mathbf{v}(t)\|^4,$$

that we integrate over $[0, s]$, $s \in [0, T]$, so that

$$\nu\|\nabla\mathbf{v}(s)\|^2 \leq \nu\|\nabla\mathbf{v}_0\|^2 + Fs + \frac{1}{C\alpha}\int_0^s \|\nabla\mathbf{v}(t)\|^4\, dt.$$

Lemma 4.1 is then applied on every time interval $[0, \tau]$ for $\tau < T$, with

$$f(t) = \nu\|\nabla\mathbf{v}(t)\|^2 \quad\text{and}\quad \lambda(t) = \frac{1}{C\alpha\nu}\|\nabla\mathbf{v}(t)\|^2,$$

which are both are in $L^1(0,\tau) \cap L^\infty(0,\tau)$ and $g(t) = \nu\|\nabla\mathbf{v}_0\|^2 + Ft$ which is a non decreasing function. These lead to

$$\nu\|\nabla\mathbf{v}(s)\|^2 \leq (\nu\|\nabla\mathbf{v}_0\|^2 + Fs)\exp\left\{\frac{1}{C\alpha\nu}\int_0^s \|\nabla\mathbf{v}(t)\|^2\, dt\right\},$$

and yields (4.3) by using (4.2). Therefore, the inequality (4.4) is deduced from (4.6) combined with (4.2)-(4.3). $\qquad\square$

4.2 Existence and uniqueness of regular-weak solutions

We start by giving the definition of a "regular-weak solution" to the generalized Navier-Stokes-Voigt system (4.1). This definition is based on Lemma 4.2. We say "weak solution" since it is given by a weak formulation, "regular" since, because of Lemma 4.2, we will search for a solution in $L^\infty(0, T; V) \cap W^{1,2}(0, T; V_{1/2})$. This space is considerably smaller than that involved in "standard" Leray-Hopf weak solutions to the Navier-Stokes equations (2.1) that are just in $L^\infty(0, T; H) \cap L^2(0, T; V)$. As we shall see, regular weak solutions are unique and satisfy the energy equality, a fact which is still not known about Leray-Hopf weak solutions to the NSE.

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Definition 4.1. *We say that a function* $\mathbf{v} \in L^\infty(0,T;V) \cap W^{1,2}(0,T;V_{1/2})$ *is a regular-weak solution of the initial boundary value problem* (4.1) *if it holds true that*

$$\frac{d}{dt}\Big[(\mathbf{v},\boldsymbol{\phi}) + \alpha(\ell D\mathbf{v}, D\boldsymbol{\phi})\Big] + \nu(\nabla\mathbf{v},\nabla\boldsymbol{\phi}) + ((\mathbf{v}\cdot\nabla)\,\mathbf{v},\boldsymbol{\phi}) = \langle\mathbf{f},\boldsymbol{\phi}\rangle \quad \forall\boldsymbol{\phi}\in V,$$

in the sense of $\mathcal{D}'(0,T)$ *and the initial datum is attained at least in the sense of* $V_{1/2}$, *that is*

$$\lim_{t\to 0^+}\|\mathbf{v}(t)-\mathbf{v}_0\|_{V_{1/2}} = 0.$$

The main theorem we prove is the following one, showing the well-posedness of the system, globally in time. To fix the ideas and for the simplicity, we stay in a usual weak solutions framework by taking the source term $\mathbf{f} = \mathbf{f}(t)$ in the space[5] $L^2(0,T;H^{-1/2}(\Omega)^3)$. However, many variants can be considered, starting with $\mathbf{f} \in L^2(0,T;V'_{1/2})$, or $\mathbf{f}(t) = \mathbf{f} \in L^2(\Omega)^3$ following the previous subsection, which does not change too much. An interesting case would be $\mathbf{f} \in L^2_{uloc}(\mathbb{R}^+;V'_{1/2})$, for which additional work remains to be done in the context of the long-time behavior (see [4]).

Theorem 4.1. *Let be given* $\mathbf{v}_0 \in V$ *and* $\mathbf{f} \in L^2(0,T;H^{-1/2}(\Omega)^3)$. *Then, there exists a unique regular-weak solution of the initial boundary value problem* (4.1) *in* $[0,T]$, *which satisfies the energy (of the model) equality for all* $t \geq 0$,

$$(4.7)\qquad E(t)(\alpha,\ell) + \nu\int_0^t \|\nabla\mathbf{v}(s)\|^2\,ds = E(0)(\alpha,\ell) + \int_0^t \langle\mathbf{f}(s),\mathbf{v}(s)\rangle\,ds.$$

where $E(t)(\alpha,\ell) := \frac{1}{2}\left(\|\mathbf{v}(t)\|^2 + \alpha\|\sqrt{\ell}D\mathbf{v}(t)\|^2\right)$.

Proof. The proof follows by a standard Faedo-Galerkin approximation with suitable a-priori estimates, compactness argument, and interpolation results. It is divided into the following four steps:

1) Construction of approximate solutions, locally in time;

2) Uniform estimates;

3) Passing to the limit in the equations;

4) Energy balance and uniqueness.

Step 1. Construction of approximate solutions, locally in time. Let $\{\boldsymbol{\psi}_n\}_n \subset \mathcal{V}$ be a Hilbert basis of V which we can suppose, without lack of generality, to be orthonormal in H as well as orthogonal in V. We look for approximate Galerkin functions

$$\mathbf{v}^n(t,x) = \sum_{j=1}^n c_{jn}(t)\,\boldsymbol{\psi}_j(x) \qquad \text{for } n\in\mathbb{N},$$

which has to solve the generalized Navier-Stokes-Voigt equations projected over $\mathbf{W}_n = \mathrm{Span}(\boldsymbol{\psi}_1,\ldots,\boldsymbol{\psi}_n)$, that is to solve

$$\frac{d}{dt}\Big[(\mathbf{v}^n,\boldsymbol{\psi}_m) + \alpha(\ell D\mathbf{v}^n, D\boldsymbol{\psi}_m)\Big] + \nu(\nabla\mathbf{v}^n,\nabla\boldsymbol{\psi}_m) + ((\mathbf{v}^n\cdot\nabla)\,\mathbf{v}^n,\boldsymbol{\psi}_m) = \langle\mathbf{f},\boldsymbol{\psi}_m\rangle,$$

$$(\mathbf{v}^n(0),\boldsymbol{\psi}_m) = (\mathbf{v}_0,\boldsymbol{\psi}_m),$$

---

[5] Recall that $H^{-1/2}(\Omega) = [H_0^{1/2}(\Omega)]'$ and be aware that $H^{-1/2}(\Omega) \subsetneq [H_{00}^{1/2}(\Omega)]'$ with strict inclusion, see Lions-Magenes [21].

for $m = 1, \ldots, n$. The above problem is a Cauchy problem for a system of n-ordinary differential equations in the coefficients $c_{nm}(t)$. We define the following quantities for $j, l, m = 1, \ldots, n$:

$$\alpha_{jm} := \alpha(\ell D\boldsymbol{\psi}_j, D\boldsymbol{\psi}_m), \qquad \beta_{jm} := \nu(\nabla\boldsymbol{\psi}_j, \nabla\boldsymbol{\psi}_m),$$
$$\gamma_{jlm} := ((\boldsymbol{\psi}_j \cdot \nabla)\,\boldsymbol{\psi}_l, \boldsymbol{\psi}_m), \qquad f_m(t) := \langle \mathbf{f}(t), \boldsymbol{\psi}_m \rangle.$$

We have a non-homogeneous system of ordinary differential equations with constant coefficients (which we write with the convention of summation over repeated indices)

$$(\delta_{jm} + \alpha_{jm})c'_{jn}(t) + \beta_{jm}c_{jn}(t) + \gamma_{jlm}c_{jn}(t)c_{ln}(t) = f_m(t), \qquad m = 1, \ldots, n,$$

where δ_{ij} denotes the standard Kronecker delta. The initial condition is $c_{jn}(0) = (\mathbf{v}_0, \boldsymbol{\psi}_j)$. The above system is not in "normal form" and in order to obtain a system for which we can apply the Cauchy-Lipschitz Theorem, we have to show that the matrix $(\delta_{jm} + \alpha_{jm})$ can be inverted. Hence, since we work in a finite dimensional spaces it is enough to show that its kernel contains only the null vector. So let $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$ be such that

$$(\delta_{jm} + \alpha_{jm})\,\xi_j = 0.$$

Multiplying the above equation by ξ_m and summing also over $m = 1, \ldots, n$ leads to

$$0 = \|\boldsymbol{\xi}\|^2 + \alpha(\ell\phi, \phi) = \|\boldsymbol{\xi}\|^2 + \alpha(\sqrt{\ell}\phi, \sqrt{\ell}\phi) \geq \|\boldsymbol{\xi}\|^2 \quad \text{with} \quad \phi := \sum_{j=1}^{n} \xi_j D\boldsymbol{\psi}_j,$$

where the last inequality holds true due to the facts that $\alpha > 0$ and $\ell(\mathbf{x}) \geq 0$. Hence, this implies that $\boldsymbol{\xi} \equiv \mathbf{0}$, and consequently the matrix $(\delta_{jm} + \alpha_{jm})$ can be inverted and we denoting by $\mathcal{A} = \mathcal{A}_{jm}$ its inverse. This allows us, after multiplication by $\mathcal{A}$, to rewrite the Galerkin system of ODEs for the coefficients c_{jn} as follows

$$c'_{jn}(t) + \beta'_{jm}c_{jn}(t) + \gamma'_{jlm}c_{jn}(t)c_{ln}(t) = f'_m(t),$$

for appropriate $\beta'_{jm}, \gamma'_{jlm}, f'_m(t)$ and to apply the basic theory of ordinary differential equations. Note that the coefficient from the right-hand side $f'_m(t) = A_{km}\langle \mathbf{f}(t), \boldsymbol{\psi}_k \rangle$ is not continuous but just $L^2(0, T)$, hence one has to resort to an extension of the Cauchy-Lipschitz theorem, with absolutely continuous functions, under Carathéodory hypotheses (see Walter [31]).

Since the system for the coefficients $c_{jn}(t)$ is nonlinear (quadratic) we obtain that there exists a unique solution $c_{jn}(t) \in W^{1,2}(0, T_n)$, for some $0 < T_n \leq T$.

Step 2. Estimates. By taking $\mathbf{v}^n$ as test function, one gets the identity

$$(4.8) \qquad \frac{1}{2}\frac{d}{dt}\left(\|\mathbf{v}^n(t)\|^2 + \alpha\|\sqrt{\ell}D\mathbf{v}^n(t)\|^2\right) + \nu\|\nabla\mathbf{v}^n(t)\|^2 = \langle \mathbf{f}, \mathbf{v}^n \rangle,$$

from which it follows

$$\frac{d}{dt}\left(\|\mathbf{v}^n(t)\|^2 + \alpha\|\sqrt{\ell}D\mathbf{v}^n(t)\|^2\right) + \nu\|\nabla\mathbf{v}^n(t)\|^2 \leq \frac{C_P}{\nu}\|\mathbf{f}\|_{-1/2,2}^2,$$

where $C_P = C_P(\Omega)$ is the Poincaré-type constant such that

$$\|u\|_{1/2,2}^2 \leq C_P\|\nabla u\|^2 \qquad \forall\, u \in H_0^1(\Omega).$$

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Hence, integrating over $(0, t)$ for $t < T_n$ we get

$$(4.9) \qquad E^n(t)(\alpha, \ell) + \nu \int_0^t \|\nabla \mathbf{v}^n(s)\|^2 \, ds \leq E^n(0)(\alpha, \ell) + \frac{C_P}{\nu} \int_0^t \|\mathbf{f}(s)\|_{-1/2,2}^2 \, ds.$$

where $E^n(t)(\alpha, \ell) := \|\mathbf{v}^n(t)\|^2 + \alpha \|\sqrt{\ell} D\mathbf{v}^n(t)\|^2$. Next, we observe that since $\mathbf{v}^n(0) \to \mathbf{v}_0$ in V and $0 \leq \ell \in C(\overline{\Omega})$, then it holds

$$\alpha \|\sqrt{\ell} D\mathbf{v}^n(0)\|^2 \leq \alpha \max_{x \in \overline{\Omega}} \ell(\mathbf{x}) \|\nabla \mathbf{v}^n(0)\|^2 \leq \alpha \max_{\mathbf{x} \in \overline{\Omega}} \ell(\mathbf{x}) \|\nabla \mathbf{v}_0\|^2,$$

which holds being $\mathbf{v}^n(0)$ the orthogonal projection of $\mathbf{v}_0$. This shows that, under the given assumptions on $\mathbf{v}_0$ and $\mathbf{f}$, the r.h.s of (4.9) can be bounded independently of $n \in \mathbb{N}$ and consequently, a standard continuation argument proves in fact that $T_n = T$. Moreover, it also holds

$$(4.10) \qquad \mathbf{v}^n \in L^\infty(0, T; H) \cap L^2(0, T; V) \quad \text{and} \quad \sqrt{\ell} D\mathbf{v}^n \in L^\infty(0, T; L^2(\Omega)^9),$$

with norms bounded uniformly in $n \in \mathbb{N}$. Therefore, according to Theorem 3.1, we also obtain

$$(4.11) \quad \|\mathbf{v}^n(t)\|^2 + \|\mathbf{v}^n(t)\|_{V_{1/2}}^2 + \int_0^t \|\nabla \mathbf{v}^n(s)\|^2 \, ds \leq C \left[\int_0^t \|\mathbf{f}(s)\|_{-1/2,2}^2 ds + \|\mathbf{v}_0\|_{1,2}^2 \right],$$

for a constant C depending on ν, α, ℓ and Ω. In addition, this inequality proves that

$$\mathbf{v}^n \in L^\infty(0, T; V_{1/2}),$$

with bounds independent of $n \in \mathbb{N}$.

In order to give a proper meaning to the time derivative, we now use as test function $\mathbf{v}_t^n$, which is allowed, since it vanishes at the boundary and it is divergence-free. We get

$$(4.12) \qquad \|\mathbf{v}_t^n(t)\|^2 + \alpha \|\sqrt{\ell} D\mathbf{v}_t^n(t)\|^2 + \frac{\nu}{2} \frac{d}{dt} \|\nabla \mathbf{v}^n(t)\|^2 = (\mathbf{f}, \mathbf{v}_t^n) - ((\mathbf{v}^n \cdot \nabla) \mathbf{v}^n, \mathbf{v}_t^n).$$

We estimate the r.h.s of (4.12), thanks to the Cauchy-Schwarz, Hölder, Young and Sobolev inequalities, which give that for all $\epsilon > 0$ there exists $C_\epsilon > 0$ such that

$$|(\mathbf{f}, \mathbf{v}_t^n)| \leq C_\epsilon \|\mathbf{f}\|_{-1/2,2}^2 + \epsilon \|\mathbf{v}_t^n\|_{V_{1/2}}^2,$$

and

$$|((\mathbf{v}^n \cdot \nabla) \mathbf{v}^n, \mathbf{v}_t^n)| \leq \|\mathbf{v}^n\|_6 \|\nabla \mathbf{v}^n\| \|\mathbf{v}_t^n\|_3 \leq C \|\nabla \mathbf{v}^n\|^2 \|\mathbf{v}_t^n\|_{V_{1/2}} \leq C_\epsilon \|\nabla \mathbf{v}^n\|^4 + \epsilon \|\mathbf{v}_t^n\|_{V_{1/2}}^2.$$

By the above inequalities we can absorb terms in the l.h.s, to obtain

$$C \|\mathbf{v}_t^n(t)\|_{V_{1/2}}^2 + \frac{\nu}{2} \frac{d}{dt} \|\nabla \mathbf{v}^n(t)\|^2 \leq C_\epsilon \left[\|\mathbf{f}(t)\|_{-1/2,2}^2 + \|\nabla \mathbf{v}^n(t)\|^4 \right],$$

for some $C_\epsilon = C(\ell, \alpha, \Omega)$. Integrating over $[0, s]$ for $s \in [0, T]$, one obtains, using the fact that $\mathbf{v}^n(0)$ is the projection of $\mathbf{v}_0$

$$(4.13) \quad C \int_0^s \|\mathbf{v}_t^n(t)\|_{V_{1/2}}^2 \, dt + \frac{\nu}{2} \|\nabla \mathbf{v}^n(s)\|^2 \leq \frac{\nu}{2} \|\nabla \mathbf{v}(0)\|^2 + C_\epsilon \int_0^s \|\mathbf{f}(t)\|_{-1/2,2}^2 \, dt$$
$$+ C_\epsilon \int_0^s \|\nabla \mathbf{v}^n(t)\|^4 \, dt,$$

and in particular,

$$\frac{\nu}{2}\|\nabla \mathbf{v}^n(s)\|^2 \le \frac{\nu}{2}\|\nabla \mathbf{v}_0\|^2 + C_\epsilon \int_0^s \|\mathbf{f}(t)\|^2_{-1/2,2}\, dt + C_\epsilon \int_0^s \|\nabla \mathbf{v}^n(t)\|^4\, dt.$$

We apply the Gronwall's lemma 4.1 to get

$$(4.14) \quad \frac{\nu}{2}\|\nabla \mathbf{v}^n(s)\|^2 \le \left(\frac{\nu}{2}\|\nabla \mathbf{v}_0\|^2 + C_\epsilon \int_0^s \|\mathbf{f}(t)\|^2_{-1/2,2}\, dt\right)\exp\left\{C_\epsilon \int_0^s \|\nabla \mathbf{v}^n(t)\|^2\, dt\right\},$$

and the r.h.s of (4.14) is bounded uniformly in n due the a priori estimate (4.10). This proves that
$$\mathbf{v}^n \in L^\infty(0,T;V),$$

from which we also deduce by using (4.13) that

$$(4.15) \qquad \mathbf{v}_t^n \in L^2(0,T;V_{1/2}), \quad \text{and therefore by } (4.10) \quad \mathbf{v}^n \in W^{1,2}(0,T;V_{1/2}),$$

with uniform bounds in $n \in \mathbb{N}$. Beside estimates in $V_{1/2}$, it is important to stress that with the same calculations, starting from (4.12) as in the proof of Lemma 4.2, we also have
$$\sqrt{\ell}D\mathbf{v}_t^n \in L^2(0,T;L^2(\Omega)^9),$$

again with uniform bound in $n \in \mathbb{N}$.

Step 3. Passing to the limit in the approximate equations. By the uniform bounds above and standard compactness results there exists $\mathbf{v} \in W^{1,2}(0,T;V_{1/2}) \cap L^\infty(0,T;V)$ and a sub-sequence (relabelled as $\mathbf{v}^n$) such that

$$\begin{cases} \mathbf{v}^n \overset{*}{\rightharpoonup} \mathbf{v} & \text{in} \quad L^\infty(0,T;V), \\ \sqrt{\ell}D\mathbf{v}^n \overset{*}{\rightharpoonup} \sqrt{\ell}D\mathbf{v} & \text{in} \quad L^\infty(0,T;L^2(\Omega)^9), \\ \mathbf{v}^n \rightharpoonup \mathbf{v} & \text{in} \quad L^p(0,T;V) \quad \text{for all } 1 < p < \infty, \\ \mathbf{v}_t^n \rightharpoonup \mathbf{v}_t & \text{in} \quad L^2(0,T;V_{1/2}), \\ \sqrt{\ell}D\mathbf{v}_t^n \rightharpoonup \sqrt{\ell}D\mathbf{v}_t & \text{in} \quad L^2(0,T;L^2(\Omega)^9), \end{cases}$$

To get strong convergence in appropriate spaces, we use the Aubin-Lions compactness lemma (see Temam [29]) with the triple

$$V \hookrightarrow V_{3/4} \hookrightarrow V_{1/2},$$

where $V_{3/4} = [V,H]_{3/4}$, each embedding being dense and continuous. Moreover, since Ω is bounded by the Rellich-Kondrachov Theorem, these embeddings are also compact. Therefore, the sequence $(\mathbf{v}^n)_{n\in\mathbb{N}}$ is (pre)compact in $L^2(0,T;V_{3/4})$ and (up to a sub-sequence)

$$\mathbf{v}^n \to \mathbf{v} \quad \text{in} \quad L^2(0,T;V_{3/4}),$$

which implies in particular strong convergence in $L^2(0,T;L^4(\Omega)^3)$. By standard results this allows to pass to the limit in the weak formulation, showing that indeed $\mathbf{v}$ is a regular-weak solution. We skip the details. It remains to check the initial data. The weak convergence implies that for $0 \le t \le T$

$$\|\mathbf{v}(t)\|^2 + \alpha\|\sqrt{\ell}D\mathbf{v}(t)\|^2 + \nu \int_0^t \|\nabla \mathbf{v}(s)\|^2\, ds \le \|\mathbf{v}(0)\|^2 + \alpha\|\sqrt{\ell}D\mathbf{v}(0)\|^2 + \int_0^t \langle \mathbf{f}(s),\mathbf{v}(s)\rangle\, ds.$$

Observe that the above inequality is obtained from (4.8), after integration in time and passing to the limit. The inequality comes from the fact that $\nabla \mathbf{v}^n \rightharpoonup \nabla \mathbf{v}$ in $L^2(0,T;L^2(\Omega)^9)$, and we have to consider the inferior limit of the norms of $\nabla \mathbf{v}^n$. In addition, we observe that since $\nabla \mathbf{v}^n(0) \to \nabla \mathbf{v}_0$ in $L^2(\Omega)$, we can suppose –up to a further sub-sequence– that $\nabla \mathbf{v}^n(0,\mathbf{x}) \to \nabla \mathbf{v}_0(\mathbf{x})$ a.e. $\mathbf{x} \in \Omega$, hence using the boundedness of ℓ and Lebesgue dominated convergence, we have

$$\|\sqrt{\ell}D\mathbf{v}^n(0)\|^2 \to \|\sqrt{\ell}D\mathbf{v}_0\|^2,$$

showing also that the initial datum is assumed strongly at the initial time.

Step 4. Energy balance and uniqueness. We start with the energy balance (4.7). To this end one has first to justify the use of $\mathbf{v}$ as test function. From the results above, we deduce that $\mathbf{v} \otimes \mathbf{v} \in L^\infty(0,T;L^3(\Omega)^9)$ which yields in particular $(\mathbf{v}\cdot\nabla)\mathbf{v} \in L^2(0,T;V')$ and $\langle (\mathbf{v}\cdot\nabla)\mathbf{v},\mathbf{v}\rangle = 0$ according to standard results. From there, the relevant point is to check that for any $s \in [0,T]$:

$$(4.16) \qquad \int_0^s (\ell D\mathbf{v}_t, D\mathbf{v})\,dt = \frac{1}{2}\|\sqrt{\ell}D\mathbf{v}(s)\|^2 - \frac{1}{2}\|\sqrt{\ell}D\mathbf{v}_0\|^2,$$

since all other terms are well-behaved due to the available regularity of $\mathbf{v}$. However, $\sqrt{\ell}D\mathbf{v}, \sqrt{\ell}D\mathbf{v}_t \in L^2(0,T;L^2(\Omega)^9)$. Therefore, by identifying $L^2(\Omega)^9$ with its dual space, we naturally have

$$\langle \ell D\mathbf{v}_t, D\mathbf{v}\rangle = \langle \sqrt{\ell}D\mathbf{v}_t, \sqrt{\ell}D\mathbf{v}\rangle = \frac{1}{2}\frac{d}{dt}\|\sqrt{\ell}D\mathbf{v}\|^2,$$

hence (4.16) and then (4.7) follows.

Moreover, this result allows us also to prove uniqueness of regular-weak solutions. In fact, if $\mathbf{v}_1$ and $\mathbf{v}_2$ are solutions corresponding to the same initial datum and same external force, taking the difference and testing (by the above argument this is fully justified) with $\mathbf{V} = \mathbf{v}_1 - \mathbf{v}_2$ one obtains the following differential equality for the difference for any $t \in [0,T]$:

$$\|\mathbf{V}(t)\|^2 + \alpha\|\sqrt{\ell}D\mathbf{V}(t)\|^2 + \nu\int_0^t \|\nabla\mathbf{V}\|^2\,ds = -\int_0^t\int_\Omega (\mathbf{V}\cdot\nabla)\,\mathbf{v}_2\cdot\mathbf{V}\,d\mathbf{x}ds.$$

Hence, by the usual Sobolev inequalities

$$\|\mathbf{V}(t)\|^2 + \alpha\|\sqrt{\ell}D\mathbf{V}(t)\|^2 + \nu\int_0^t \|\nabla\mathbf{V}\|^2\,ds \le \frac{\nu}{2}\int_0^t \|\nabla\mathbf{V}\|^2\,ds + \frac{C}{\nu}\int_0^t \|\nabla\mathbf{v}_2\|^4\|\mathbf{V}\|^2\,ds,$$

and since $\mathbf{V}(0) = \mathbf{0}$ the Gronwall's lemma shows that $\mathbf{V} \equiv \mathbf{0}$, due to the fact that

$$\nabla\mathbf{v}_2 \in L^\infty(0,T;L^2(\Omega)^9) \subset L^4(0,T;L^2(\Omega)^9).$$

$\square$

Remark 4.3. *The pressure is not involved in Definition 4.1. However, let $(\mathbf{v}_0,\mathbf{f})$ be given as in Theorem 4.1 and let $\mathbf{v}$ be the corresponding regular-weak solution. Then, by the De Rham theorem, we easily deduce the existence of $p \in \mathcal{D}'(0,T;L^2(\Omega)/\mathbb{R})$ such that $(\mathbf{v},p)$ satisfies system (4.1) in the sense of the distributions. The regularity of the pressure is probably even better than that, but this point, which is inessential for the results of this paper, remains to be investigated.*

Remark 4.4. *Definition [4.1] is equivalent to the following: The field $\mathbf{v}$ is a regular-weak solution to* [(4.1)] *if:*

1. $\mathbf{v} \in W^{1,2}(0,T;V_{1/2}) \cap L^\infty(0,T;V)$, $\sqrt{\ell}D\mathbf{v}_t \in L^2(Q_T)^9$,

2. *for all $\mathbf{w} \in L^2(0,T;V)$, $\forall\, s < T$:*

$$\int_0^s (\mathbf{v}_t, \mathbf{w})\, dt + \alpha \int_0^s (\sqrt{\ell}D\mathbf{v}_t, \sqrt{\ell}D\mathbf{w})\, dt - \int_0^s \int_\Omega \mathbf{v} \otimes \mathbf{v} : \nabla\mathbf{w}\, d\mathbf{x}dt$$
$$+ \nu \int_0^s \int_\Omega \nabla\mathbf{v} : \nabla\mathbf{w}\, d\mathbf{x}dt = \int_0^s \langle \mathbf{f}, \mathbf{w} \rangle\, dt,$$

3. $\lim_{t\to 0^+} \|\mathbf{v}(t) - \mathbf{v}_0\|_{V_{1/2}} = 0$.

Once the above results of existence and uniqueness have been proved for the generalized Navier-Stokes-Voigt equations, it is straightforward to prove the same also for the model [(1.1)] with an additional turbulent viscosity ν_{turb} which is non-negative and bounded. We do not reproduce here the proof, which follows the same steps of Theorem [4.1] since the additional term with the turbulent viscosity ν_{turb} neither improves the a priori estimates neither creates additional problems when taking weak limits of the approximate solutions.

Theorem 4.2. *Let $\nu_{\mathrm{turb}} \in L^\infty([0,\infty[\times\Omega)$ such that $\nu_{\mathrm{turb}} \geq 0$ a.e. in $[0,\infty[\times\Omega$ and let and $\mathbf{v}_0 \in V$ and $\mathbf{f} \in L^2(0,T;H^{-1/2}(\Omega)^3)$ be given. Then, the system*

$$(4.17) \quad \begin{cases} \mathbf{v}_t - \alpha\nabla\cdot(\ell\, D\mathbf{v}_t) + (\mathbf{v}\cdot\nabla)\,\mathbf{v} - \nu\Delta\mathbf{v} - \nabla\cdot(\nu_{\mathrm{turb}}D\mathbf{v}) + \nabla p = \mathbf{f} & in\ (0,T)\times\Omega, \\ \nabla\cdot\mathbf{v} = 0 & in\ (0,T)\times\Omega, \\ \mathbf{v}_{|\Gamma} = 0 & on\ (0,T)\times\Gamma, \\ \mathbf{v}_{|t=0} = \mathbf{v}_0 & in\ \Omega. \end{cases}$$

has a unique regular-weak solution that satisfies the energy balance[6] (equality)

$$E(t)(\alpha,\ell) + \int_0^t \|(2\nu + \nu_{\mathrm{turb}})^{1/2}D\mathbf{v}(s)\|^2\, ds = E(0)(\alpha,\ell) + \int_0^t \langle \mathbf{f}(s), \mathbf{v}(s) \rangle\, ds,$$

where again $E(t)(\alpha,\ell) = \dfrac{1}{2}\left(\|\mathbf{v}(t)\|^2 + \alpha\|\sqrt{\ell}D\mathbf{v}(t)\|^2\right)$.

5 Turbulent Voigt model involving the TKE

In this section we consider the generalized Voigt model with an additional turbulent viscosity $\nu_{\mathrm{turb}} = \nu_{\mathrm{turb}}(k)$, coupled with the evolution equation for the turbulent kinetic energy k. In particular, we prove a compactness result which allows then to prove in an easy way existence of weak solutions.

5.1 A compactness Lemma

We consider a family of models as in [(4.17)], associated with different realizations of the turbulent viscosity and study the behavior of the solutions, under rather mild conditions on the given additional viscosities.

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[6]Remind that since $\nabla\cdot\mathbf{v} = 0$, then $\Delta\mathbf{v} = 2\nabla\cdot D\mathbf{v}$. Therefore, $\langle -\nu\Delta\mathbf{v} + \nabla\cdot(\nu_{\mathrm{turb}}D\mathbf{v}), \mathbf{w}\rangle = ((2\nu + \nu_{\mathrm{turb}})D\mathbf{v}, D\mathbf{w})$.

To this end let be given a family of turbulent viscosities $(\nu_{\text{turb}}^n)_{n \in \mathbb{N}}$ such that

$$\forall\, n \geq 0, \qquad \nu_{\text{turb}}^n \in L^\infty([0, \infty[\times\Omega), \qquad \nu_{\text{turb}}^n \geq 0 \ \text{a.e. in } [0, \infty[\times\Omega.$$

Let $\mathbf{v}_0 \in V$ and $\mathbf{f} \in L^2(0, T; H^{-1/2}(\Omega)^3)$. Let $(\mathbf{v}^n, p^n)$ finally denote the distributional solution to the following boundary initial value problem

(5.1)
$$\begin{cases}
\mathbf{v}_t^n - \alpha \nabla \cdot (\ell\, D\mathbf{v}_t^n) + (\mathbf{v}^n \cdot \nabla)\, \mathbf{v}^n - \nu \Delta \mathbf{v}^n - \nabla \cdot (\nu_{\text{turb}}^n D\mathbf{v}^n) + \nabla p^n = \mathbf{f} & \text{in } (0, T) \times \Omega, \\
\nabla \cdot \mathbf{v}^n = 0 & \text{in } (0, T) \times \Omega, \\
\mathbf{v}_{|\Gamma}^n = 0 & \text{on } (0, T) \times \Gamma, \\
\mathbf{v}_{|t=0}^n = \mathbf{v}_0 & \text{in } \Omega,
\end{cases}$$

and such that $\mathbf{v}^n \in W^{1,2}(0, T; V_{1/2}) \cap L^\infty(0, T; V)$, is a regular-weak solution to (5.1).

Remark 5.1. *In this section $\mathbf{v}^n$ denotes the velocity associated to the turbulent viscosity ν_{turb}^n, namely (5.1), and not a Galerkin approximation as in the previous section. We keep the same notation for the simplicity, since there is no risk of confusion.*

Concerning the behavior as $n \to +\infty$ of the family of solutions $(\mathbf{v}^n)_{n \in \mathbb{N}}$ we have the following lemma.

Lemma 5.1. *Assume that the sequence $(\nu_{\text{turb}}^n)_{n \in \mathbb{N}}$ is uniformly bounded in $L^\infty([0, \infty[\times\Omega)$ and converges almost everywhere to ν_{turb} in $Q_\infty := [0, \infty[\times\Omega$.*
Then, it follows that:

1) *The sequence $(\mathbf{v}^n)_{n \in \mathbb{N}}$ weakly converges in $W^{1,2}(0, T; V_{1/2}) \cap L^p(0, T; V)$, for all $p < \infty$, to the regular-weak solution $\mathbf{v}$ of the limit problem*

(5.2)
$$\begin{cases}
\mathbf{v}_t - \alpha \nabla \cdot (\ell\, D\mathbf{v}_t) + (\mathbf{v} \cdot \nabla)\, \mathbf{v} - \nu \Delta \mathbf{v} - \nabla \cdot (\nu_{\text{turb}} D\mathbf{v}) + \nabla p = \mathbf{f} & \text{in } (0, T) \times \Omega, \\
\nabla \cdot \mathbf{v} = 0 & \text{in } (0, T) \times \Omega, \\
\mathbf{v}_{|\Gamma} = 0 & \text{on } (0, T) \times \Gamma, \\
\mathbf{v}_{|t=0} = \mathbf{v}_0 & \text{in } \Omega.
\end{cases}$$

2) *The sequence $(\nu_{\text{turb}}^n |D\mathbf{v}^n|^2)_{n \in \mathbb{N}}$ converges in $L^1(Q_T)$ and the sense of measures to $\nu_{\text{turb}} |D\mathbf{v}|^2$ in Q_T, which means that*

(5.3) $$\forall\, \varphi \in C(\overline{Q}_T), \quad \int_0^T \int_\Omega \nu_{\text{turb}}^n |D\mathbf{v}^n|^2 \varphi\, d\mathbf{x}dt \xrightarrow{n \to \infty} \int_0^T \int_\Omega \nu_{\text{turb}} |D\mathbf{v}|^2 \varphi\, d\mathbf{x}dt.$$

Proof. In order to simplify the notation we extract sub-sequences, without changing the notation. However, by the uniqueness result of Theorem 4.1, we finally get convergence for the whole sequence because of the uniqueness of solutions to the limit problem.

1) The estimates (4.11), (4.13), (4.14), and (4.15) we previously derived for the Galerkin approximations, hold true also for the present sequence $(\mathbf{v}^n)_{n \in \mathbb{N}}$. Therefore, the sequence $(\mathbf{v}^n)_{n \in \mathbb{N}}$ is weakly pre-compact in $W^{1,2}(0, T; V_{1/2}) \cap L^p(0, T; V)$, for all $p < \infty$, as well as strongly pre-compact in $L^2(0, T; V_{3/4})$, by Aubin-Lions Lemma. Therefore, it weakly converges (up to a sub-sequence) to some $\mathbf{v}$ in $W^{1,2}(0, T; V_{1/2}) \cap L^p(0, T; V)$, strongly in $L^2(0, T; V_{3/4})$. In particular, the strong convergence holds in $L^2(0, T; L^4(\Omega))$, and also in $L^4(0, T; L^4(\Omega))$ by usual arguments.

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Passing to the limit in the equations satisfied by $\mathbf{v}^n$ is then straightforward (cf. Theorem 4.1), except in the eddy viscosity term. To this end, for any given $\mathbf{w} \in L^2(0, T; V)$, we can write

$$\langle -\nabla \cdot (\nu_{\text{turb}}^n D\mathbf{v}^n), \mathbf{w} \rangle = \int_0^T \int_\Omega \nu_{\text{turb}}^n D\mathbf{v}^n : D\mathbf{w} \, d\mathbf{x} ds = \int_0^T \int_\Omega D\mathbf{v}^n : \nu_{\text{turb}}^n D\mathbf{w} \, d\mathbf{x} ds.$$

Next, as $(\nu_{\text{turb}}^n)_{n \in \mathbf{N}}$ is bounded in $L^\infty(Q_T)$, we have the following estimate

$$\text{for a.e } (t, \mathbf{x}) \in Q_T \qquad |\nu_{\text{turb}}^n(t, \mathbf{x}) D\mathbf{w}(t, \mathbf{x})| \leq \sup_{n \in \mathbf{N}} \|\nu_{\text{turb}}^n\|_{L_{t,\mathbf{x}}^\infty} |D\mathbf{w}(t, \mathbf{x})| \in L^2(Q_T).$$

On the other hand, according to the *a.e* convergence of ν_{turb}^n, it also follows that

$$\nu_{\text{turb}}^n D\mathbf{w} \to \nu_{\text{turb}} D\mathbf{w} \quad \text{a.e in } Q_T.$$

Then, by the Lebesgue dominated convergence theorem, one has the strong convergence

$$\nu_{\text{turb}}^n D\mathbf{w} \to \nu_{\text{turb}} D\mathbf{w} \quad \text{in } L^2(Q_T).$$

From the weak convergence of $\mathbf{v}^n$ in $L^2(0, T; V)$ we have

$$D\mathbf{v}^n \rightharpoonup D\mathbf{v} \quad \text{in } L^2(Q_T).$$

The convergence of the eddy viscosity term follows then from

$$\langle -\nabla \cdot (\nu_{\text{turb}}^n D\mathbf{v}^n), \mathbf{w} \rangle = \int_0^T \int_\Omega D\mathbf{v}^n : \nu_{\text{turb}}^n D\mathbf{w} \, d\mathbf{x} ds \xrightarrow{n \to \infty} \int_0^T \int_\Omega D\mathbf{v} : \nu_{\text{turb}} D\mathbf{w} \, d\mathbf{x} ds$$
$$= \langle -\nabla \cdot (\nu_{\text{turb}} D\mathbf{v}), \mathbf{w} \rangle,$$

As a consequence, $\mathbf{v}$ is indeed a regular-weak solution to (5.2) on $[0, T]$, for all positive T, and by the uniqueness of the solution, all the sequence does converge.

2) We split the proof of this property into three steps:

i) Weak convergence of the sequence $((2\nu + \nu_{\text{turb}}^n)^{1/2} D\mathbf{v}^n)_{n \in \mathbf{N}}$ to $(2\nu + \nu_{\text{turb}})^{1/2} D\mathbf{v}$ in $L^2(Q_T)$;

ii) Strong convergence of the same sequence by the "energy method";

iii) Proof of the convergence in measures from (5.3).

i) This steps is very similar to point 1). In fact, we already proved that the sequence $((2\nu + \nu_{\text{turb}}^n)^{1/2} D\mathbf{v}^n)_{n \in \mathbf{N}}$ is bounded in $L^2(Q_T)^9$, uniformly in $n \in \mathbf{N}$. Moreover, we already know that $D\mathbf{v}^n \rightharpoonup D\mathbf{v}$ in $L^2(Q_T)$. Let us define

$$\mathbf{A}_n := (2\nu + \nu_{\text{turb}}^n)^{1/2} D\mathbf{v}^n \qquad \text{and} \qquad \mathbf{A} := (2\nu + \nu_{\text{turb}})^{1/2} D\mathbf{v}.$$

We aim to prove that $\mathbf{A}_n \rightharpoonup \mathbf{A}$ in $L^2(Q_T)^9$. To do so, let us fix $\mathbf{w} \in L^2(0, T; V)$. By the hypothesis of a.e. convergence on $(\nu_{\text{turb}}^n)_{n \in \mathbf{N}}$ it follows that

$$(2\nu + \nu_{\text{turb}}^n)^{1/2} D\mathbf{w} \xrightarrow{n \to \infty} (2\nu + \nu_{\text{turb}})^{1/2} D\mathbf{w} \quad \text{a.e. in } Q_T.$$

Moreover, one has also

$$\left| (2\nu + \nu_{\text{turb}}^n(t, \mathbf{x}))^{1/2} D\mathbf{w}(t, \mathbf{x}) \right| \leq \left(2\nu + \sup_n \|\nu_{\text{turb}}^n\|_{L_{t,\mathbf{x}}^\infty} \right)^{1/2} |D\mathbf{w}(t, \mathbf{x})| \in L^2(Q_T).$$

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Therefore, again by Lebesgue's theorem we obtain the strong convergence

$$(2\nu + \nu_{\text{turb}}^n)^{1/2} D\mathbf{w} \xrightarrow{n\to\infty} (2\nu + \nu_{\text{turb}})^{1/2} D\mathbf{w} \quad \text{in } L^2(Q_T).$$

As before weak L^2-convergence of $D\mathbf{v}^N$ implies

$$\int_0^T \int_\Omega (2\nu + \nu_{\text{turb}}^n)^{1/2} D\mathbf{v}^n : D\mathbf{w}\, d\mathbf{x}dt = \int_0^T \int_\Omega (2\nu + \nu_{\text{turb}}^n)^{1/2} D\mathbf{w} : D\mathbf{v}^n\, d\mathbf{x}dt$$

$$\xrightarrow{n\to\infty} \int_0^T \int_\Omega (2\nu + \nu_{\text{turb}})^{1/2} D\mathbf{w} : D\mathbf{v}\, d\mathbf{x}dt = \int_0^T \int_\Omega (2\nu + \nu_{\text{turb}})^{1/2} D\mathbf{v} : D\mathbf{w}\, d\mathbf{x}dt,$$

yielding the desired weak convergence.

ii) *Energy method.* We now prove the strong L^2-convergence of the sequence $(\mathbf{A}_n)_{n\in\mathbf{N}}$ to $\mathbf{A}$. To do so, we use the energy method (see [7, 18]), based on the energy (equality) balance (4.7) satisfied by both $\mathbf{v}^n$ and $\mathbf{v}$, with dissipative terms from eddy viscosity terms

$$\int_0^t \int_\Omega \nu_{\text{turb}}^n |D\mathbf{v}^n|^2 \quad \text{and} \quad \int_0^t \int_\Omega \nu_{\text{turb}} |D\mathbf{v}|^2 \qquad t \in [0, T],$$

in the corresponding equations. Observe that at this stage is very important to have the energy balance satisfied with an equality, instead of the inequality, as it holds for the (unregularized) Navier-Stokes equations. For an overview and recent results on the possible energy equality for incompressible fluids, see [2].

According to the notations introduced in Theorem 4.1, we rewrite the energy balances as follows, for all $t < T$,

(5.4)
$$E(t)(\alpha, \ell) + \int_0^t \int_\Omega |\mathbf{A}|^2\, d\mathbf{x}ds = \int_0^t \langle \mathbf{f}, \mathbf{v}\rangle\, ds + E(0)(\alpha, \ell),$$
$$E^n(t)(\alpha, \ell) + \int_0^t \int_\Omega |\mathbf{A}_n|^2\, d\mathbf{x}ds = \int_0^t \langle \mathbf{f}, \mathbf{v}^n\rangle\, ds + E(0)(\alpha, \ell),$$

A critical tool in the energy method is that of integrating over $[0, T]$, with the respect to the time variable, both the equations in (5.4) and to perform then an integration by parts. This yields the following two equalities

(5.5)
$$\int_0^T E(t)(\alpha, \ell)\, dt + \int_0^T \int_\Omega (T - t)|\mathbf{A}|^2\, d\mathbf{x}dt = \int_0^T \int_0^t \langle \mathbf{f}, \mathbf{v}\rangle\, dsdt + TE(0)(\alpha, \ell),$$
$$\int_0^T E^N(t)(\alpha, \ell)\, dt + \int_0^T \int_\Omega (T - t)|\mathbf{A}_n|^2\, d\mathbf{x}dt = \int_0^T \int_0^t \langle \mathbf{f}, \mathbf{v}^n\rangle\, dsdt + TE(0)(\alpha, \ell).$$

Arguing with the usual compactness tools as developed in the proof of Theorems 4.1 4.2 we obtain that

$$\int_0^T \|\mathbf{v}^n(t)\|^2 dt \xrightarrow{n\to\infty} \int_0^T \|\mathbf{v}(t)\|^2\, dt,$$
$$\int_0^T \int_0^t \langle \mathbf{f}, \mathbf{v}^n\rangle\, dsdt \xrightarrow{n\to\infty} \int_0^T \int_0^t \langle \mathbf{f}, \mathbf{v}\rangle\, dsdt$$
$$TE^N(0)(\alpha, \ell) \xrightarrow{n\to\infty} TE(0)(\alpha, \ell).$$

Therefore, by using the integrated energy equalities (5.5), we also get by comparison

$$\int_0^T \int_\Omega \left[\alpha\ell|D\mathbf{v}^n|^2 + (T - t)|\mathbf{A}_n|^2\right]\, d\mathbf{x}dt \xrightarrow{n\to\infty} \int_0^T \int_\Omega \left[\alpha\ell|D\mathbf{v}|^2 + (T - t)|\mathbf{A}|^2\right]\, d\mathbf{x}dt.$$

Let now $\mathbf{B}_n, \mathbf{B}$ be defined as follows

$$\mathbf{B}_n := (\alpha\ell + (T-t)(2\nu + \nu_{\mathrm{turb}}^n))^{1/2}D\mathbf{v}^n \quad \text{and} \quad \mathbf{B} := (\alpha\ell + (T-t)(2\nu + \nu_{\mathrm{turb}}))^{1/2}D\mathbf{v}.$$

The information coming from the convergence of the integrated energy equalities can be rewritten as follows

$$\|\mathbf{B}_N\|^2 \xrightarrow{n\to\infty} \|\mathbf{B}\|^2.$$

Next, the assumptions on the function $\ell(\mathbf{x})$ imply also that, for a given $\mathbf{w} \in L^2(0,T;V)$,

$$\left|(\alpha\ell(\mathbf{x}) + (T-t)(2\nu + \nu_{\mathrm{turb}}^n(t,\mathbf{x})))^{1/2} D\mathbf{w}(t,\mathbf{x})\right|$$

$$\leq \left(\alpha \max_{\mathbf{x}\in\Omega} \ell + T(2\nu + \sup_n \|\nu_{\mathrm{turb}}^n\|_{L_{t,\mathbf{x}}^\infty})\right)^{1/2} |D\mathbf{w}(t,\mathbf{x})| \in L^2(Q_T).$$

and, by using the same argument as before, this implies the strong L^2-convergence of $(\alpha\ell(\mathbf{x}) + (T-t)(2\nu + \nu_{\mathrm{turb}}^n(t,\mathbf{x})))^{1/2} D\mathbf{w}(t,\mathbf{x})$. The weak $L^2(Q_T)$-convergence of $D\mathbf{v}^N$ to $D\mathbf{v}$ implies the weak $L^2(Q_T)$ convergence $\mathbf{B}_N \rightharpoonup \mathbf{B}$. This together with the convergence of the norms $\|\mathbf{B}_N\|$ implies that

$$\mathbf{B}_n \xrightarrow{n\to\infty} \mathbf{B} \quad \text{in } L^2(Q_T)^9.$$

Next, we observe that, for all $t < T$ and for all $\mathbf{x} \in \Omega$ we can write

$$\mathbf{A}_n = (2\nu + \nu_{\mathrm{turb}}^n))^{1/2}D\mathbf{v}^n = (\alpha\ell + (T-t)(2\nu + \nu_{\mathrm{turb}}^n))^{1/2}D\mathbf{v}^n \frac{(2\nu + \nu_{\mathrm{turb}}^n))^{1/2}}{(\alpha\ell + (T-t)(2\nu + \nu_{\mathrm{turb}}^n))^{1/2}}$$

$$= \mathbf{B}_n \frac{(2\nu + \nu_{\mathrm{turb}}^n))^{1/2}}{(\alpha\ell + (T-t)(2\nu + \nu_{\mathrm{turb}}^n))^{1/2}}$$

and for all $T' < T$ it holds

$$\frac{(2\nu + \nu_{\mathrm{turb}}^n))^{1/2}}{(\alpha\ell + (T-t)(2\nu + \nu_{\mathrm{turb}}^n))^{1/2}} \xrightarrow{n\to\infty} \frac{(2\nu + \nu_{\mathrm{turb}}))^{1/2}}{(\alpha\ell + (T-t)(2\nu + \nu_{\mathrm{turb}}))^{1/2}} \qquad \text{a.e. in } Q_{T'}.$$

The above point-wise convergence and the L^2-strong convergence of $\mathbf{B}_N$ allow us to prove then

$$\mathbf{A}_n \to \mathbf{B} \frac{(2\nu + \nu_{\mathrm{turb}}))^{1/2}}{(\alpha\ell + (T-t)(2\nu + \nu_{\mathrm{turb}}))^{1/2}} = \mathbf{A} \qquad \text{in } L^2(Q_{T'}), \quad \forall T' < T.$$

As T can be any positive time, this concludes this step.

iii) *Proof of* (5.3). The same argument used in the above step can be also used to show that

$$\sqrt{\nu_{\mathrm{turb}}^n}D\mathbf{v}^n \to \sqrt{\nu_{\mathrm{turb}}}D\mathbf{v} \qquad \text{in } L^2(Q_T),$$

simply writing

$$\sqrt{\nu_{\mathrm{turb}}^n}D\mathbf{v}^n = (\alpha\ell + (T-t)(2\nu + \nu_{\mathrm{turb}}^n))^{1/2}D\mathbf{v}^n \frac{\sqrt{\nu_{\mathrm{turb}}^n}}{(\alpha\ell + (T-t)(2\nu + \nu_{\mathrm{turb}}^n))^{1/2}}$$

as the right-hand side converges in $L^2(Q_T)$ as $n \to +\infty$ to

$$\mathbf{B} \frac{\sqrt{\nu_{\mathrm{turb}}^n}}{(\alpha\ell + (T-t)(2\nu + \nu_{\mathrm{turb}}^n))^{1/2}} = \sqrt{\nu_{\mathrm{turb}}}D\mathbf{v}.$$

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This shows that

$$(5.6) \qquad \int_{Q_T} \nu_{\text{turb}}^n(t,\mathbf{x})|D\mathbf{v}^N(t,\mathbf{x})|^2 \, d\mathbf{x}dt \to \int_{Q_T} \nu_{\text{turb}}(t,\mathbf{x})|D\mathbf{v}(t,\mathbf{x})|^2 \, d\mathbf{x}dt.$$

On one hand, we deduce from the above that the sequence $(\sqrt{\nu_{\text{turb}}^n}D\mathbf{v}^n)_{n\in\mathbb{N}}$ weakly converges to $\sqrt{\nu_{\text{turb}}}D\mathbf{v}$ in $L^2(Q_T)$. On the other hand, the corresponding sequence of the norms also converges by (5.6). Hence $(\sqrt{\nu_{\text{turb}}^n}D\mathbf{v}^n)_{n\in\mathbb{N}}$ strongly converges to $\sqrt{\nu_{\text{turb}}}D\mathbf{v}$ in $L^2(Q_T)$, and the conclusion of part 2 of Lemma 5.1 easily follows. □

5.2 Application to the NSTKE-Voigt model

We now apply the existence result for the Generalized Voigt model, together with the compactness lemma, to study the Voigt model coupled with the equation of the turbulent kinetic energy. The NSTKE-Voigt model is in fact obtained by coupling the turbulent Navier-Stokes-Voigt equation with the equation for the TKE, following the law (2.4), which gives the following initial boundary value problem

$$(5.7) \quad \begin{cases} \mathbf{v}_t - \alpha\nabla\cdot(\ell\,D\mathbf{v}_t) + (\mathbf{v}\cdot\nabla)\,\mathbf{v} - \nu\Delta\mathbf{v} - \nabla\cdot(\nu_{\text{turb}}(k)D\mathbf{v}) + \nabla p = \mathbf{f}, & (i) \\ \nabla\cdot\mathbf{v} = 0, & (ii) \\ \mathbf{v}_{|\Gamma} = 0, & (iii) \\ \mathbf{v}_{|t=0} = \mathbf{v}_0, & (iv) \\ k_t + \mathbf{v}\cdot\nabla k - \nabla\cdot(\mu_{\text{turb}}(k)\nabla k) = \nu_{\text{turb}}(k)|D\mathbf{v}|^2 - (\ell+\eta)^{-1}k\sqrt{|k|}, & (v) \\ k_{|\Gamma} = 0, & (vi) \\ k_{|t=0} = k_0. & (vii) \end{cases}$$

This system calls for two comments:

1) According to Lemma 5.1, we know how to deal with bounded eddy viscosities and the argument does not work with general viscosities. This is why we cannot take the unbounded law (2.4). We replace it, as often done with this class of problems, by

   $$(5.8) \qquad \nu_{\text{turb}}(k) = \ell\,T_{M'}(\sqrt{|k|}), \qquad M' \in \mathbb{N}$$

   where T_M is the usual "truncation function" at height M, for a given large $M \in \mathbb{N}$, which is defined by

   $$T_M(x) := \begin{cases} x & \text{if } |x| \le M, \\ M\dfrac{x}{|x|} & \text{if } |x| > M, \end{cases}$$

   for all $x \in \mathbb{R}$. The eddy viscosity (5.8) will fit with that considered in Lemma 5.1. Similarly, we assume that the diffusion coefficient for the turbulent kinetic energy satisfies

   $$(5.9) \qquad \mu_{\text{turb}}(k) = C\ell\,T_{M''}(\sqrt{|k|}),$$

   for some dimensionless constant C and another $M'' \in \mathbb{N}$.

2) Usually, the dissipation term in the r.h.s of the equation for k is considered as $\varepsilon := \ell^{-1}k\sqrt{|k|}$. Unfortunately, due to the degeneration of ℓ at the boundary Γ, there could be further issues when passing to the limit in this term. As a precaution, we have approximated it by $\varepsilon = (\ell+\eta)^{-1}k\sqrt{|k|}$ where $\eta > 0$ is a small parameter. We have not studied the behavior of the solutions when $\eta \to 0$, yet.

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The main existence result we are able to prove for the NSTKE-Voigt system is the following.

Theorem 5.1. *Let be given* $\mathbf{v}_0 \in V$, $\mathbf{f} \in L^2(0, T; H^{-1/2}(\Omega)^3)$ *and* $0 \leq k_0 \in L^1(\Omega)$. *Assume that* ν_{turb} *and* μ_{turb} *are given by* (5.8) *and* (5.9). *Then. there exists* $(\mathbf{v}, k)$ *such that:*

1. *The vector field* $\mathbf{v}$ *verifies*

$$\mathbf{v} \in L^\infty(0, T; V) \cap W^{1,2}(0, T; V_{1/2}),$$

   and it is a regular-weak solution to the subsystem [(5.7)-(i)-(ii)-(iii)-(iv))],

2. *The scalar field* k *verifies*

$$k \in L^\infty(0, T; L^1(\Omega)), \quad k \in \bigcap_{1<p<5/4} L^p(0, T; W^{1,p}(\Omega)) = K_{5/4},$$

   and it is solution of the subsystem [(5.7)-(v)-(vi)-(vii))] in the sense of the distribution in Q_T. *Moreover,* $k \geq 0$ *a.e. in* Q_T.

Proof. System (5.7) is very close to that studied for in [7, Chapter 8]. Therefore, we only indicate the changes in the proof of existence, without giving full details, which can be easily filled by the reader. The main difference is given by the result of the compactness Lemma 5.1, which is essential to the proof. The further (compared to the previously studied systems) regularity enforced by the generalized Voigt term is the key to prove the existence results for the full NSTKE model.

One technical issue is due to the quadratic source term $\nu_{\mathrm{turb}}(k)|D\mathbf{v}|^2$ in the TKE equation, which is *a priori* in $L^1(Q_T)$ and not better. Hence, we have to deal with a generalized Navier-Stokes-Voigt system with a right-hand side in $L^1(Q_T)$, for which the theory cannot be directly handled. To overcome this fact, we truncate this term as well as the initial data, leading to the following regularized (truncated) system:

$$(5.10) \quad \begin{cases} \mathbf{v}_t - \alpha\nabla \cdot (\ell\, D\mathbf{v}_t) + (\mathbf{v}\cdot\nabla)\,\mathbf{v} - \nu\Delta\mathbf{v} - \nabla\cdot(\nu_{\mathrm{turb}}(k)D\mathbf{v}) + \nabla p = \mathbf{f}, & (i) \\ \nabla\cdot\mathbf{v} = 0, & (ii) \\ \mathbf{v}_{|\Gamma} = 0, & (iii) \\ \mathbf{v}_{|t=0} = \mathbf{v}_0, & (iv) \\ k_t + \mathbf{v}\cdot\nabla k - \nabla\cdot(\mu_{\mathrm{turb}}(k)\nabla k) = T_M(\nu_{\mathrm{turb}}(k)|D\mathbf{v}|^2) - (\ell+\eta)^{-1}k\sqrt{|k|}, & (v) \\ k_{|\Gamma} = 0, & (vi) \\ k_{|t=0} = T_M(k_0). & (vii) \end{cases}$$

The proof of existence for the approximate (truncated) system is now done my means of suitable applications of the Schauder fixed-point theorem.

As first step we fix $\widetilde{k} \in L^2(0, T; H_0^1(\Omega)) \cap L^\infty(0, T; L^2(\Omega))$, let $\widetilde{\mathbf{v}} = \mathbf{v}(\widetilde{k})$ be the unique regular-weak solution to

$$\begin{cases} \widetilde{\mathbf{v}}_t - \alpha\nabla\cdot(\ell\, D\widetilde{\mathbf{v}}_t) + (\widetilde{\mathbf{v}}\cdot\nabla)\,\widetilde{\mathbf{v}} - \nu\Delta\widetilde{\mathbf{v}} - \nabla\cdot(\nu_{\mathrm{turb}}(\widetilde{k})D\widetilde{\mathbf{v}}) + \nabla p = \mathbf{f}, \\ \nabla\cdot\widetilde{\mathbf{v}} = 0, \\ \widetilde{\mathbf{v}}_{|\Gamma} = 0, \\ \widetilde{\mathbf{v}}_{|t=0} = \mathbf{v}_0, \end{cases}$$

which is the subsystem [(5.10)-(i)-(ii)-(iii)-(iv))] with $\nu_{\text{turb}}(k)$ is replaced by $\nu_{\text{turb}}(\widetilde{k})$. The existence of $\widetilde{\mathbf{v}}$ follows from Theorems 4.1-4.2. The next step is to analyze the equation for k, considering

$$(5.11) \quad \begin{cases} k_t + (\widetilde{\mathbf{v}} \cdot \nabla) \, k - \nabla \cdot (\mu_{\text{turb}}(k)\nabla k) = T_n(\nu_{\text{turb}}(k)|D\widetilde{\mathbf{v}}|^2) - (\ell + \eta)^{-1}k\sqrt{|k|}, \\ k_{|\Gamma} = 0, \\ k_{|t=0} = T_n(k_0), \end{cases}$$

which is a non linear parabolic equation with both coefficients and a source term smooth enough to allow application of the standard variational theory. The existence of a weak solution $k \in L^2(0,T; H_0^1(\Omega)) \cap L^\infty(0,T; L^2(\Omega))$ to Problem (5.11), it is easily proved and, in addition, it follows $k_t \in L^2(0,T; H^{-1}(\Omega))$.

The solution of full system (5.10) –with $M \in \mathbb{N}$ fixed– can be obtained by finding a fixed point of the map

$$\widetilde{k} \to k,$$

by means of the Schauder fixed point theorem as in [7], as follows by the boundedness in appropriate spaces. Observe that this will not ensure uniqueness of the solution.

To solve the full system (5.7) (with truncated viscosities but with the L^1-right-hand-side) we iteratively construct the sequence

$$(k^n)_{n\in\mathbb{N}},$$

in the following way: we start from $k^0 \equiv 0$ and $\mathbf{v}^0$ the corresponding solution of the [(5.7)-(i)-(ii)-(iii)-(iv))], with $\nu_{\text{turb}} = \nu_{\text{turb}}(0)$. Then, we iteratively construct the sequence of solutions along the following Picard iterative scheme, which is suitable by the uniqueness result: For $n \geq 1$, let be given

$$(\mathbf{v}^{n-1}, k^{n-1}) \in L^\infty(0,T;V) \cap W^{1,2}(0,T;V_{1/2}) \times L^2(0,T;H_0^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega)).$$

Then, the couple

$$(\mathbf{v}^n, k^n) \in L^\infty(0,T;V) \cap W^{1,2}(0,T;V_{1/2}) \times L^2(0,T;H_0^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega)),$$

is defined as the solution of the following system
(5.12)
$$\begin{cases} \mathbf{v}_t^n - \alpha\nabla \cdot (\ell \, D\mathbf{v}_t^n) + (\mathbf{v}^n \cdot \nabla) \, \mathbf{v}^n - \nu\Delta\mathbf{v}^n - \nabla \cdot (\nu_{\text{turb}}(k^n)D\mathbf{v}^n) + \nabla p^n = \mathbf{f}, \\ \nabla \cdot \mathbf{v}^n = 0, \\ \mathbf{v}_{|\Gamma}^n = 0, \\ \mathbf{v}_{|t=0}^n = \mathbf{v}_0, \\ k_t^n + (\mathbf{v}^{n-1} \cdot \nabla) \, k^n - \nabla \cdot (\mu_{\text{turb}}(k^n)\nabla k^n) = T_n(\nu_{\text{turb}}(k^n)|D\mathbf{v}^{n-1}|^2) - (\ell + \eta)^{-1}k^n\sqrt{|k^n|}, \\ k_{|\Gamma}^n = 0, \\ k_{|t=0}^n = T_n(k_0). \end{cases}$$

By using the estimates from [7, Chapter 8] that, up to a sub-sequence,

$$(5.13) \quad \begin{cases} k^n \rightharpoonup k & \text{in} \quad L^q(0,T;W_0^{1,q}(\Omega)) \qquad \text{for all } 1 \leq q < 5/4, \\ k_t^n \rightharpoonup k_t & \text{in} \quad L^q(0,T;W^{-1,q}(\Omega)) \qquad \text{for all } 1 \leq q < 5/4, \\ k^n \to k & \text{in} \quad L^q(Q_T) \qquad \text{for all } 1 \leq q < 29/14 \quad \text{and a.e. in } Q_T. \end{cases}$$

Next, we observe that $x \mapsto \ell \, T_{M'}\big(\sqrt{|x|}\big)$ is a continuous function over $\mathbb{R}$, we have that $\nu_{\text{turb}}^n = \nu_{\text{turb}}(k^n) \to \nu_{\text{turb}} = \nu_{\text{turb}}(k)$ a.e. in Q_T. Next, since $\ell \in C^1(\overline{\Omega})$, we also have

$$0 \leq \nu_{\text{turb}} \leq M'\|\ell\|_\infty,$$

showing that $(\nu_{\mathrm{turb}}^n)_{n\in\mathbb{N}}$ verifies all the requirements of Lemma 5.1, by (5.8) and (5.13). Therefore, $\mathbf{v}^n \to \mathbf{v} = \mathbf{v}(k)$, the corresponding regular-weak solution to the subsystem [(5.12), (i), (ii), (iii), (iv))]. Notice that thanks to the uniqueness of the limit, all the sequence does converges.

Passing to the limit in the equation for k follows what is done in [7, Chapter 8], except about the quadratic source term, that needs to be reconsidered. Always by Lemma 5.1, $\nu_{\mathrm{turb}}(k^n)|D\mathbf{v}^n|^2 \to \nu_{\mathrm{turb}}(k)|D\mathbf{v}|^2$ in $L^1(Q_T)$ and in the sense of measures. Finally, according to Lemma 5.2 below, we deduce that $T_n(\nu_{\mathrm{turb}}(k^n)|D\mathbf{v}^n|^2) \to \nu_{\mathrm{turb}}(k)|D\mathbf{v}|^2$ in $L^1(Q_T)$ up to a sub-sequence, the conclusion being from this point straightforward (cf. Lemma 5.2 below), which ends the proof. □

For the reader's convenience we report the following result about convergence of truncated sequences.

Lemma 5.2. *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $L^1(Q_T)$ that converges to f in $L^1(Q_T)$. Then from the sequence $(T_n(f_n))_{n\in\mathbb{N}}$ we can extract a sub-sequence that still converges to f in $L^1(Q_T)$.*

Proof. Since $f_n \to f$ in $L^1(Q_T)$, we can find a sub-sequence (still denoted by f_n) and a function $F \in L^1(Q_T)$ such that

  i) $f_n(t,\mathbf{x}) \to f(t,\mathbf{x})$ for a.e. $(t,\mathbf{x}) \in Q_T$;

  ii) $|f_n(t,\mathbf{x})| \leq F(t,\mathbf{x})$ for all $n \in \mathbb{N}$ and for a.e. $(t,\mathbf{x}) \in Q_T$.

Hence, the sequence $g_n(t,\mathbf{x}) = T_n(f_n(t,\mathbf{x}))$ satisfies the following,

  1) The sequence $g_n(t,\mathbf{x}) = T_n(f_n(t,\mathbf{x}))$ converges to $f(t,\mathbf{x})$ for a.e. $(t,\mathbf{x}) \in Q_T$ and $f(t,\mathbf{x})$ is a.e. finite. This follows since a.e. for fixed $(t_0,\mathbf{x}_0)$ it holds $f_n(t_0,\mathbf{x}_0) < f(t_0,\mathbf{x}_0) + 1$ for all $n > n_0(t_0,\mathbf{x}_0) \in \mathbb{N}$ hence $g_n(t_0,\mathbf{x}_0) = f_n(t_0,\mathbf{x}_0)$ for all $n > \max\{n_0, f(t_0,\mathbf{x}_0) + 1\}$;

  2) The sequence $g_n(t,\mathbf{x}) = T_n(f_n(t,\mathbf{x}))$ is uniformly equi-absolutely integrable, that is for all $\epsilon > 0$ there exists $\delta > 0$ such that for all measurable $H \subset Q_T$ such that $\mathrm{meas}(H) < \delta$, it holds

$$\sup_{n\in\mathbb{N}} \int_H |T_n(f_n(t,\mathbf{x}))|\,d\mathbf{x} \leq \sup_{n\in\mathbb{N}} \int_H |f_n(t,\mathbf{x})|\,d\mathbf{x} \leq \int_H |F(t,\mathbf{x})|\,d\mathbf{x} < \epsilon,$$

  the last inequality being valid since $F \in L^1(Q_T)$.

The hypotheses of Vitali convergence theorem are satisfied, hence $g_n \to g$ in $L^1(Q_T)$. □

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